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A TEXT-BOOK OF GEOMETRY.

REVISED EDITION.

BY

G. A. WENTWORTH, A.M.,
AUTHOR OF A SERIES OF TEXT-BOOKS IN MATHEMATICS.

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BOSTON, U.S.A.:
GINN & COMPANY, PUBLISHERS.
1893.

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PREFACE.

MOST persons do not possess, and do not easily acquire, the power of abstraction requisite for apprehending geometrical conceptions, and for keeping in mind the successive steps of a continuous argument. Hence, with a very large proportion of beginners in Geometry, it depends mainly upon the *form* in which the subject is presented whether they pursue the study with indifference, not to say aversion, or with increasing interest and pleasure.

In compiling the present treatise, the author has kept this fact constantly in view. All unnecessary discussions and scholia have been avoided; and such methods have been adopted as experience and attentive observation, combined with repeated trials, have shown to be most readily comprehended. No attempt has been made to render more intelligible the simple notions of position, magnitude, and direction, which every child derives from observation; but it is believed that these notions have been limited and defined with mathematical precision.

A few symbols, which stand for words and not for operations, have been used, but these are of so great utility in giving *style* and *perspicuity* to the demonstrations that no apology seems necessary for their introduction.

Great pains have been taken to make the page attractive. The figures are large and distinct, and are placed in the middle of the page, so that they fall directly under the eye in immediate connection with the corresponding text. The *given* lines of the figures are full lines, the lines employed as *aids* in the demonstrations are short-dotted, and the *resulting* lines are long-dotted.

In each proposition a concise statement of what is given is printed in one kind of type, of what is required in another, and the demonstration in still another. The reason for each step is indicated in small type between that step and the one following, thus preventing the necessity of interrupting the process of the argument by referring to a previous section. The number of the section, however, on which the reason depends is placed at the side of the page. The constituent parts of the propositions are carefully marked. *Moreover, each distinct assertion in the demonstrations and each particular direction in the construction of the figures, begins a new line; and in no case is it necessary to turn the page in reading a demonstration.*

This arrangement presents obvious advantages. The pupil perceives at once what is given and what is required, readily refers to the figure at every step, becomes perfectly familiar with the language of Geometry, acquires facility in simple and accurate expression, rapidly *learns to reason*, and lays a foundation for completely establishing the science.

Original exercises have been given, not so difficult as to discourage the beginner, but well adapted to afford an effectual test of the degree in which he is *mastering* the subjects of his reading. Some of these exercises have been placed in the early part of the work in order that the student may discover, at the outset, that to commit to memory a number of theorems and to reproduce them in an examination is a useless and pernicious labor; but to learn their uses and applications, and to acquire a readiness in exemplifying their utility is to derive the full benefit of that mathematical training which looks not so much to the *attainment of information* as to the *discipline of the mental faculties.*

G. A. WENTWORTH.

PHILLIPS EXETER ACADEMY,
1878.

TO THE TEACHER.

When the pupil is reading each Book for the first time, it will be well to let him write his proofs on the blackboard in his own language; care being taken that his language be the simplest possible, that the arrangement of work be vertical (without side work), and that the figures be accurately constructed.

This method will furnish a valuable exercise as a language lesson, will cultivate the habit of neat and orderly arrangement of work, and will allow a brief interval for deliberating on each step.

After a Book has been read in this way, the pupil should review the Book, and should be required to draw the figures free-hand. He should state and prove the propositions orally, using a pointer to indicate on the figure every line and angle named. He should be encouraged, in reviewing each Book, to do the original exercises; to state the converse of propositions; to determine from the statement, if possible, whether the converse be true or false, and if the converse be true to demonstrate it; and also to give well-considered answers to questions which may be asked him on many propositions.

The Teacher is strongly advised to illustrate, geometrically and arithmetically, the principles of limits. Thus a rectangle with a constant base b, and a variable altitude x, will afford an obvious illustration of the axiomatic truth that the product of a constant and a variable is also a variable; and that the limit of the product of a constant and a variable is the product of the constant by the limit of the variable. If x increases and approaches the altitude a as a limit, the area of the rectangle increases and approaches the area of the rectangle ab as a limit; if, however, x decreases and approaches zero as a limit, the area of the rectangle decreases and approaches zero for a limit. An arithmetical illustration of this truth may be given by multiplying a constant into the approximate values of any repetend. If, for example, we take the constant 60 and the repetend 0.3333, etc., the approximate values of the repetend will be $\frac{3}{10}$, $\frac{33}{100}$,

$\frac{333}{1000}$, $\frac{3333}{10000}$, etc., and these values multiplied by 60 give the series 18, 19.8, 19.98, 19.9998, etc., which evidently approaches 20 as a limit; but the product of 60 into $\frac{1}{3}$ (the limit of the repetend 0.333, etc.) is also 20.

Again, if we multiply 60 into the different values of the decreasing series $\frac{1}{30}$, $\frac{1}{300}$, $\frac{1}{3000}$, $\frac{1}{30000}$, etc., which approaches zero as a limit, we shall get the decreasing series 2, $\frac{1}{5}$, $\frac{1}{50}$, $\frac{1}{500}$, etc.; and this series evidently approaches zero as a limit.

In this way the pupil may easily be led to a complete comprehension of the subject of limits.

The Teacher is likewise advised to give frequent written examinations. These should not be too difficult, and sufficient time should be allowed for accurately constructing the figures, for choosing the best language, and for determining the best arrangement.

The time necessary for the reading of examination-books will be diminished by more than one-half, if the use of the symbols employed in this book be allowed.

G. A. W.

PHILLIPS EXETER ACADEMY,
1879.

NOTE TO REVISED EDITION.

The first edition of this Geometry was issued about nine years ago. The book was received with such general favor that it has been necessary to print very large editions every year since, so that the plates are practically worn out. Taking advantage of the necessity for new plates, the author has re-written the whole work; but has retained all the distinguishing characteristics of the former edition. A few changes in the order of the subject-matter have been made, some of the demonstrations have been given in a more concise and simple form than before, and the treatment of Limits and of Loci has been made as easy of comprehension as possible.

More than seven hundred exercises have been introduced into this edition. These exercises consist of theorems, loci, problems of construction, and problems of computation, *carefully graded and specially adapted to beginners*. No geometry can now receive favor unless it provides exercises for independent investigation, which must be of such a kind as to interest the student as soon as he becomes acquainted with the methods and the spirit of geometrical reasoning. The author has observed with the greatest satisfaction the rapid growth of the demand for original exercises, and he invites particular attention to the systematic and progressive series of exercises in this edition.

The part on Solid Geometry has been treated with much greater freedom than before, and the formal statement of the reasons for the separate steps has been in general omitted, for the purpose of giving a more elegant form to the demonstrations.

A brief treatise on Conic Sections (Book IX) has been prepared, and is issued in pamphlet form, at a very low price. It will also be bound with the Geometry if that arrangement is found to be generally desired.

The author takes this opportunity to express his grateful appreciation of the generous reception given to the Geometry heretofore by the great body of teachers throughout the country, and he confidently anticipates the same generous judgment of his efforts to bring the work up to the standard required by the great advance of late in the science and method of teaching.

The author is indebted to many correspondents for valuable suggestions; and a special acknowledgment is due, for criticisms and careful reading of proofs, to Messrs. C. H. Judson, of Greenville, S.C.; Samuel Hart, of Hartford, Conn.; J. M. Taylor, of Hamilton, N.Y.; W. Le Conte Stevens, of Brooklyn, N.Y.; E. R. Offutt, of St. Louis, Mo.; J. L. Patterson, of Lawrenceville, N.J.; G. A. Hill, of Cambridge, Mass.; T. M. Blakslee, Des Moines, Ia.; G. W. Sawin, of Cambridge, Mass.; and Ira M. De Long, of Boulder, Col.

Corrections or suggestions will be thankfully received.

G. A. WENTWORTH

PHILLIPS EXETER ACADEMY,
1888.

CONTENTS.

GEOMETRY.

PLANE GEOMETRY.

BOOK I. The Straight Line.

BOOK II. The Circle.

BOOK III. Proportional Lines and Similar Polygons.

BOOK IV. Areas of Polygons.

BOOK V. Regular Polygons and Circles.

SOLID GEOMETRY.

BOOK VI. Lines and Planes in Space.

BOOK VII. Polyhedrons, Cylinders, and Cones.

BOOK VIII. The Sphere.

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GEOMETRY.

DEFINITIONS.

1. If a block of wood or stone be cut in the shape represented in Fig. 1, it will have *six flat faces.*

Each face of the block is called a *surface;* and if these faces are made smooth by polishing, so that, when a straight-edge is applied to any one of them, the straight edge in every part will touch the surface, the faces are called *plane surfaces*, or *planes.*

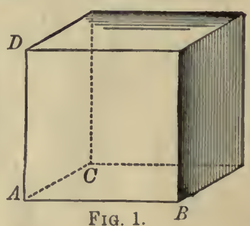


FIG. 1.

2. The *edge* in which any two of these surfaces meet is called a *line.*

3. The *corner* at which any three of these lines meet is called a *point.*

4. For computing its volume, the block is measured in *three* principal directions:

From left to right, A to B.
From front to back, A to C.
From bottom to top, A to D.

These three measurements are called the *dimensions* of the block, and are named *length*, *breadth* (or *width*), *thickness* (*height* or *depth*).

A solid, therefore, has three dimensions, length, breadth, and thickness.

5. The surface of a solid is no part of the solid. It is simply the boundary or limit of the solid. *A surface, therefore, has only two dimensions, length and breadth.* So that, if any number of flat surfaces be put together, they will coincide and form one surface.

6. A line is no part of a surface. It is simply a boundary or limit of the surface. *A line, therefore, has only one dimension, length.* So that, if any number of straight lines be put together, they will coincide and form one line.

7. A point is no part of a line. It is simply the limit of the line. *A point, therefore, has no dimension, but denotes position simply.* So that, if any number of points be put together, they will coincide and form a single point.

8. A *solid*, in common language, is a limited portion of space *filled with matter;* but in Geometry we have nothing to do with the matter of which a body is composed; we study simply its *shape* and *size;* that is, we regard a solid as a limited portion of space which may be occupied by a physical body, or marked out in some other way. Hence,

A geometrical solid is a limited portion of space.

9. It must be distinctly understood at the outset that the points, lines, surfaces, and solids of Geometry are *purely ideal*, though they can be represented to the eye in only a material way. Lines, for example, drawn on paper or on the blackboard, will have some width and some thickness, and will so far fail of being *true lines;* yet, when they are used to help the mind in reasoning, it is assumed that they represent perfect lines, without breadth and without thickness.

10. A point is *represented* to the eye by a fine dot, and named by a letter, as A (Fig. 2); a line is named by two letters, placed one at each end, as BF; a surface is represented and named by the lines which bound it, as $BCDF$; a solid is represented by the faces which bound it.

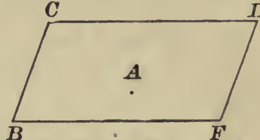


Fig. 2.

11. By supposing a solid to diminish gradually until it vanishes we may consider the vanishing point, a *point* in space, independent of a line, having *position* but *no extent.*

12. If a point moves continuously in space, its path is a line. This line may be supposed to be of *unlimited extent*, and may be considered independent of the idea of a surface.

13. A surface may be conceived as generated by a line moving in space, and as of *unlimited extent.* A surface can then be considered independent of the idea of a solid.

14. A solid may be conceived as generated by a surface in motion.

Thus, in the diagram, let the upright surface $ABCD$ move to the right to the position $EFGH$. The points A, B, C, and D will generate the lines AE, BF, CG, and DH, respectively. The lines AB, BC, CD, and AD will generate the surfaces AF, BG, CH, and AH, respectively. The surface $ABCD$ will generate the solid AG.

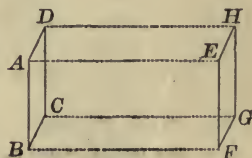


Fig. 3.

15. *Geometry* is the science which treats of *position*, *form*, and *magnitude.*

16. Points, lines, surfaces, and solids, with their relations, constitute the subject-matter of Geometry.

17. A *straight line*, or *right line*, is a line which has the same direction throughout its whole extent, as the line AB.

18. A *curved line* is a line no part of which is straight, as the line CD.

19. A *broken line* is a series of different successive straight lines, as the line EF.

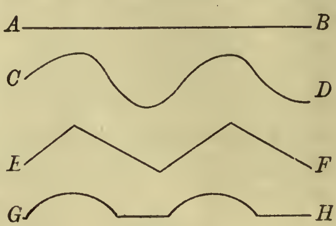


FIG. 4.

20. A *mixed line* is a line composed of straight and curved lines, as the line GH.

A straight line is often called simply a *line*, and a curved line, a *curve*.

21. A *plane surface*, or a *plane*, is a surface in which, if any two points be taken, the straight line joining these points will lie wholly in the surface.

22. A *curved surface* is a surface no part of which is plane.

23. *Figure* or *form* depends upon the relative position of points. Thus, the figure or form of a line (straight or curved) depends upon the relative position of the points in that line; the figure or form of a surface depends upon the relative position of the points in that surface.

24. With reference to *form* or *shape*, lines, surfaces, and solids are called *figures*.

With reference to *extent*, lines, surfaces, and solids are called *magnitudes*.

25. A *plane figure* is a figure all points of which are in the same plane.

26. Plane figures formed by straight lines are called *rectilinear* figures; those formed by curved lines are called *curvilinear* figures; and those formed by straight and curved lines are called *mixtilinear* figures.

27. Figures which have the same *shape* are called *similar* figures. Figures which have the same *size* are called *equivalent* figures. Figures which have the same *shape and size* are called *equal* or *congruent* figures.

28. Geometry is divided in two parts, Plane Geometry and Solid Geometry. Plane Geometry treats of figures all points of which are in the same plane. Solid Geometry treats of figures all points of which are not in the same plane.

Straight Lines.

29. Through a point an indefinite number of straight lines may be drawn. These lines will have different directions.

30. If the direction of a straight line and a point in the line are known, the position of the line is known; in other words, a straight line is *determined* if its direction and one of its points are known. Hence,

All straight lines which pass through the same point in the same direction coincide, and form but one line.

31. Between two points one, and only one, straight line can be drawn; in other words, a straight line is determined if two of its points are known. Hence,

Two straight lines which have two points common coincide throughout their whole extent, and form but one line.

32. Two straight lines can *intersect* (cut each other) in only one point; for if they had two points common, they would coincide and not intersect.

33. Of all lines joining two points the *shortest* is the straight line, and the length of the straight line is called the *distance* between the two points.

34. A straight line determined by two points is considered as prolonged indefinitely both ways. Such a line is called an *indefinite straight line.*

35. Often only the part of the line between two fixed points is considered. This part is then called a *segment* of the line.

For brevity, we say "the line AB" to designate a segment of a line limited by the points A and B.

36. Sometimes, also, a line is considered as proceeding from a fixed point and extending in only one direction. This fixed point is then called the *origin* of the line.

37. If any point C be taken in a given straight line AB, the two parts CA and CB are said to have *opposite directions* from the point C.

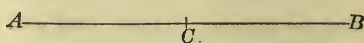


FIG. 5.

38. Every straight line, as AB, may be considered as having opposite directions, namely, from A towards B, which is expressed by saying "line AB"; and from B towards A, which is expressed by saying "line BA."

39. If the magnitude of a given line is changed, it becomes longer or shorter.

Thus (Fig. 5), by prolonging AC to B we add CB to AC, and $AB = AC + CB$. By diminishing AB to C, we subtract CB from AB, and $AC = AB - CB$.

If a given line increases so that it is prolonged by its own magnitude several times in succession, the line is *multiplied*, and the resulting line is called a *multiple* of the given line. Thus (Fig. 6), if $AB = BC = CD = DE$, then $AC = 2\,AB$, $AD = 3\,AB$, and $AE = 4\,AB$. Also, $AB = \frac{1}{2}\,AC$, $AB = \frac{1}{3}\,AD$, and $AB = \frac{1}{4}\,AE$. Hence,

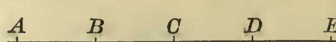


FIG. 6.

Lines of given length may be added and subtracted; they may also be multiplied and divided by a number.

Plane Angles.

40. The *opening* between two straight lines which meet is called a *plane angle.* The two lines are called the *sides*, and the point of meeting, the *vertex*, of the angle.

41. If there is but one angle at a given vertex, it is designated by a capital letter placed at the vertex, and is read by simply naming the letter; as, angle A (Fig. 7).

Fig. 7.

But when two or more angles have the same vertex, each angle is designated by three letters, as shown in Fig. 8, and is read by naming the three letters, the one at the vertex between the others. Thus, the angle DAC means the angle formed by the sides AD and AC.

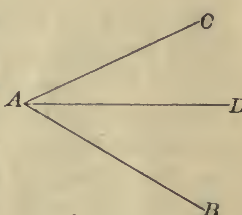


Fig. 8.

It is often convenient to designate an angle by placing a small *italic* letter between the sides and near the vertex, as in Fig. 9.

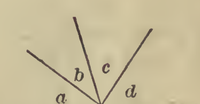


Fig. 9.

42. Two angles are *equal* if they can be made to coincide.

43. If the line AD (Fig. 8) is drawn so as to divide the angle BAC into two equal parts, BAD and CAD, AD is called the *bisector* of the angle BAC. In general, a line that divides a geometrical magnitude into two equal parts is called a bisector of it.

44. Two angles are called *adjacent* when they have the same vertex and a common side between them; as, the angles BOD and AOD (Fig. 10).

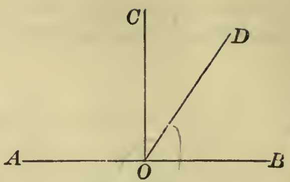


FIG. 10.

45. When one straight line stands upon another straight line and makes the *adjacent angles equal*, each of these angles is called a *right angle*. Thus, the equal angles DCA and DCB (Fig. 11) are each a right angle.

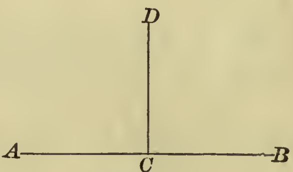


FIG. 11.

46. When the sides of an angle extend in opposite directions, so as to be in the same straight line, the angle is called a *straight angle*. Thus, the angle formed at C (Fig. 11) with its sides CA and CB extending in opposite directions from C, is a straight angle. Hence a *right angle* may be defined as *half a straight angle*.

47. A *perpendicular* to a straight line is a straight line that makes a right angle with it. Thus, if the angle DCA (Fig. 11) is a right angle, DC is perpendicular to AB, and AB is perpendicular to DC.

48. The point (as C, Fig. 11) where a perpendicular meets another line is called the *foot* of the perpendicular.

49. Every angle less than a right angle is called an *acute angle*; as, angle A.

FIG. 12.

50. Every angle greater than a right angle and less than a straight angle is called an *obtuse angle*; as, angle C (Fig. 13).

51. Every angle greater than a straight angle and less than two straight angles is called a *reflex angle;* as, angle O (Fig. 14).

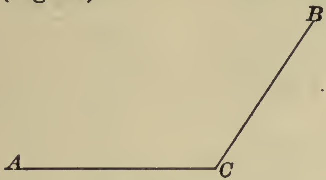


FIG. 13.

FIG. 14.

52. Acute, obtuse, and reflex angles, in distinction from right and straight angles, are called *oblique* angles; and intersecting lines that are not perpendicular to each other are called *oblique lines.*

53. When two angles have the same vertex, and the sides of the one are prolongations of the sides of the other, they are called *vertical angles.* Thus, a and b (Fig. 15) are vertical angles.

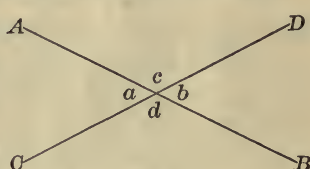


FIG. 15.

54. Two angles are called *complementary* when their sum is equal to a right angle; and each is called the *complement* of the other; as, angles DOB and DOC (Fig. 10).

55. Two angles are called *supplementary* when their sum is equal to a straight angle; and each is called the *supplement* of the other; as, angles DOB and DOA (Fig. 10).

MAGNITUDE OF ANGLES.

56. The *size* of an angle depends upon the *extent of opening* of its sides, and not upon their length. Suppose the straight

line OC to move in the plane of the paper from coincidence with OA, about the point O as a pivot, to the position OC; then the line OC describes or generates *the angle* AOC.

The *amount of rotation* of the line from the position OA to the position OC is the acute angle AOC.

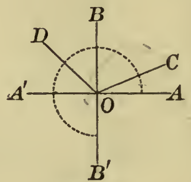


FIG. 16.

If the rotating line moves from the position OA to the position OB, perpendicular to OA, it generates the right angle AOB; if it moves to the position OD, it generates the obtuse angle AOD; if it moves to the position OA', it generates the straight angle AOA'; if it moves to the position OB', it generates the reflex angle AOB', indicated by the dotted line; and if it continues its rotation to the position OA, whence it started, it generates two straight angles.

Hence the whole angular magnitude about a point in a plane is equal to two straight angles, or four right angles; and the angular magnitude about a point on one side of a straight line drawn through that point is equal to one straight angle, or two right angles.

Angles are magnitudes that can be added and subtracted; they may also be multiplied and divided by a number.

ANGULAR UNITS.

57. If we suppose OC (Fig. 17) to turn about O from a position coincident with OA until it makes a complete revolution and comes again into coincidence with OA, it will describe *the whole angular magnitude about the point* O, while its end point C will describe a curve called a *circumference*.

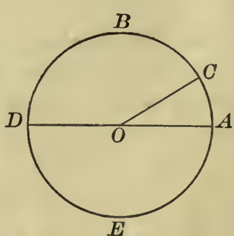


FIG. 17.

58. By adopting a suitable unit of angles we are able to express the magnitudes of angles in numbers.

If we suppose OC (Fig. 17) to turn about O from coincidence with OA until it makes *one three hundred and sixtieth* of a revolution, it generates an angle at O, which is taken as the unit for measuring angles. This unit is called a *degree.*

The degree is subdivided into sixty equal parts called *minutes*, and the minute into sixty equal parts, called *seconds.*

Degrees, minutes, and seconds are denoted by symbols. Thus, 5 degrees 13 minutes 12 seconds is written, $5^\circ\ 13'\ 12''$.

A right angle is generated when OC has made *one-fourth* of a revolution and is an angle of 90°; a straight angle is generated when OC has made *one-half* of a revolution and is an angle of 180°; and the whole angular magnitude about O is generated when OC has made a complete revolution, and contains 360°.

The natural angular unit is one complete revolution. But the adoption of this unit would require us to express the values of all angles by fractions. The advantage of using the degree as the unit consists in its convenient size, and in the fact that 360 is divisible by so many different integral numbers.

Method of Superposition.

59. The test of the equality of two geometrical magnitudes is that they coincide throughout their whole extent.

Thus, two straight lines are equal, if they can be so placed that the points at their extremities coincide. Two angles are equal, if they can be so placed that they coincide.

In applying this test of equality, we assume that a line may be moved from one place to another without altering its length; that an angle may be taken up, turned over, and put down, without altering the difference in direction of its sides.

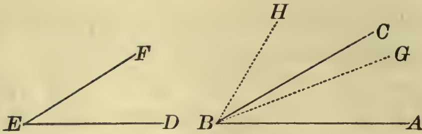


Fig. 18.

This method enables us to compare magnitudes of the same kind. Suppose we have two angles, ABC and DEF. Let the side ED be placed on the side BA, so that the vertex E shall fall on B; then, if the side EF falls on BC, the angle DEF equals the angle ABC; if the side EF falls between BC and BA in the direction BG, the angle DEF is less than ABC; but if the side EF falls in the direction BH, the angle DEF is greater than ABC.

This method enables us to add magnitudes of the same kind. Thus, if we have two straight lines AB and CD, by placing the point C on B, and keeping CD in the same direction with AB, we shall have one continuous straight line AD equal to the sum of the lines AB and CD.

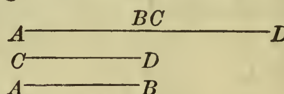


Fig. 19.

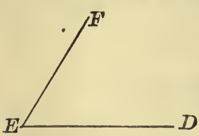


Fig. 20.

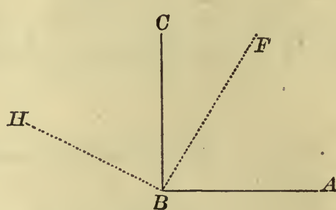


Fig. 21.

Again: if we have the angles ABC and DEF, and place the vertex E on B and the side ED in the direction of BC, the angle DEF will take the position CBH, and the angles DEF and ABC will together equal the angle ABH.

If the vertex E is placed on B, and the side ED on BA, the angle DEF will take the position ABF, and the angle FBC will be the difference between the angles ABC and DEF.

SYMMETRY.

60. Two points are said to be *symmetrical* with respect to a third point, called the *centre of symmetry*, if this third point bisects the straight line which joins them. Thus, P and P' are symmetrical with respect to C as a centre, if C bisects the straight line PP'.

FIG. 22.

61. Two points are said to be *symmetrical* with respect to a straight line, called the *axis of symmetry*, if this straight line bisects at right angles the straight line which joins them. Thus, P and P' are symmetrical with respect to XX' as an axis, if XX' bisects PP' at right angles.

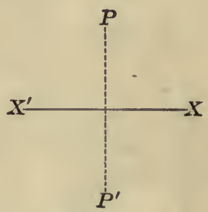


FIG. 23.

62. Two figures are said to be symmetrical with respect to a centre or an axis if every point of one has a corresponding symmetrical point in the other. Thus, if every point in the figure $A'B'C'$ has a symmetrical point in ABC, with respect to D as a centre, the figure $A'B'C'$ is symmetrical to ABC with respect to D as a centre.

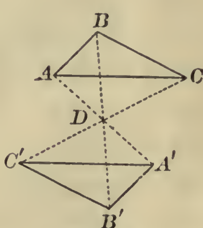


FIG. 24.

63. If every point in the figure $A'B'C'$ has a symmetrical point in ABC, with respect to XX' as an axis, the figure $A'B'C'$ is symmetrical to ABC with respect to XX' as an axis.

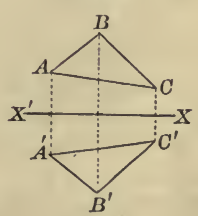


FIG. 25.

64. A figure is symmetrical with respect to a point, if the point bisects every straight line drawn through it and terminated by the boundary of the figure.

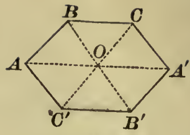


FIG. 26.

65. A plane figure is symmetrical with respect to a straight line, if the line divides it into two parts, which are symmetrical with respect to this straight line.

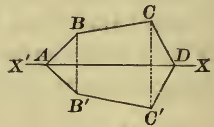


FIG. 27.

MATHEMATICAL TERMS.

66. A *proof* or *demonstration* is a course of reasoning by which the truth or falsity of any statement is logically established.

67. A *theorem* is a statement to be proved.

68. A theorem consists of two parts: the hypothesis, or that which is assumed; and the conclusion, or that which is asserted to follow from the hypothesis.

69. An *axiom* is a statement the truth of which is admitted without proof.

70. A *construction* is a graphical representation of a geometrical figure.

71. A *problem* is a question to be solved.

72. The solution of a problem consists of four parts:

(1) The *analysis*, or course of thought by which the construction of the required figure is discovered;

(2) The *construction* of the figure with the aid of ruler and compasses;

(3) The *proof* that the figure satisfies all the given conditions;

(4) The *discussion* of the limitations, which often exist, within which the solution is possible.

73. A *postulate* is a construction admitted to be possible.

74. A *proposition* is a general term for either a theorem or a problem.

75. A *corollary* is a truth easily deduced from the proposition to which it is attached.

76. A *scholium* is a remark upon some particular feature of a proposition.

77. The *converse* of a theorem is formed by interchanging its hypothesis and conclusion. Thus,

If A is equal to B, C is equal to D. (Direct.)
If C is equal to D, A is equal to B. (Converse.)

78. The *opposite* of a proposition is formed by stating the negative of its hypothesis and its conclusion. Thus,

If A is equal to B, C is equal to D. (Direct.)
If A is not equal to B, C is not equal to D. (Opposite.)

79. The converse of a truth is not *necessarily* true. Thus, Every horse is a quadruped is a true proposition, but the converse, Every quadruped is a horse, is not true.

80. *If a direct proposition and its converse are true, the opposite proposition is true; and if a direct proposition and its opposite are true, the converse proposition is true.*

81. POSTULATES.

Let it be granted —

1. That a straight line can be drawn from any one point to any other point.
2. That a straight line can be produced to any distance, or can be terminated at any point.
3. That a circumference may be described about any point as a centre with a radius of given length.

82. AXIOMS.

1. Things which are equal to the same thing are equal to each other.

2. If equals are added to equals the sums are equal.

3. If equals are taken from equals the remainders are equal.

4. If equals are added to unequals the sums are unequal, and the greater sum is obtained from the greater magnitude.

5. If equals are taken from unequals the remainders are unequal, and the greater remainder is obtained from the greater magnitude.

6. Things which are double the same thing, or equal things, are equal to each other.

7. Things which are halves of the same thing, or of equal things, are equal to each other.

8. The whole is greater than any of its parts.

9. The whole is equal to all its parts taken together.

83. SYMBOLS AND ABBREVIATIONS.

+ increased by.	⊙ circle. Ⓢ circles.
− diminished by.	Def. . . . . definition.
× multiplied by.	Ax. . . . . axiom.
÷ divided by.	Hyp. . . . hypothesis.
= is (or are) equal to.	Cor. . . . . corollary.
≎ is (or are) equivalent to.	Adj. . . . . adjacent.
> is (or are) greater than.	Iden. . . . identical.
< is (or are) less than.	Cons. . . . construction.
∴ therefore.	Sup. . . . . supplementary.
∠ angle.	Sup.-adj. supplementary-adjacent.
∠s angles.	Ext.-int. exterior-interior.
⊥ perpendicular.	Alt.-int. alternate-interior.
⊥s perpendiculars.	Ex. . . . . exercise.
∥ parallel.	rt. . . . . . right.
∥s parallels.	st. . . . . . straight.
△ triangle.	Q.E.D. . . . quod erat demonstrandum, *which was to be proved.*
△s triangles.	Q.E.F. . . . quod erat faciendum, *which was to be done.*
▱ parallelogram.	
▱s parallelograms.	

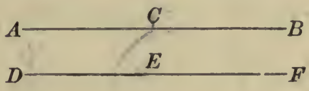
PLANE GEOMETRY.

BOOK I.

THE STRAIGHT LINE.

PROPOSITION I. THEOREM.

84. *All straight angles are equal.*



Let $\angle BCA$ and $\angle FED$ be any two straight angles.

To prove $\angle BCA = \angle FED$.

Proof. Apply the $\angle BCA$ to the $\angle FED$, so that the vertex C shall fall on the vertex E, and the side CB on the side EF.

Then CA will coincide with ED,

(*because BCA and FED are straight lines and have two points common*).

Therefore the $\angle BCA$ is equal to the $\angle FED$. § 59

85. COR. 1. *All right angles are equal.*

86. COR. 2. *The angular units, degree, minute, and second, have constant values.*

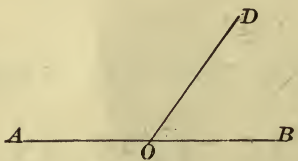
87. COR. 3. *The complements of equal angles are equal.*

88. COR. 4. *The supplements of equal angles are equal.*

89. COR. 5. *At a given point in a given straight line one perpendicular, and only one, can be erected.*

PROPOSITION II. THEOREM.

90. *If two adjacent angles have their exterior sides in a straight line, these angles are supplements of each other.*



Let the exterior sides OA and OB of the adjacent $\angle s$ AOD and BOD be in the straight line AB.

To prove $\angle s$ AOD and BOD supplementary.

Proof.	AOB is a straight line.	Hyp.
	$\therefore$ the $\angle\, AOB$ is a st. $\angle$.	§ 46
But	the $\angle s$ $AOD + BOD =$ the st. $\angle\, AOB$.	Ax. 9
	$\therefore$ the $\angle s$ AOD and BOD are supplementary.	§ 55

Q. E. D.

91. SCHOLIUM. Adjacent angles that are supplements of each other are called *supplementary-adjacent angles.*

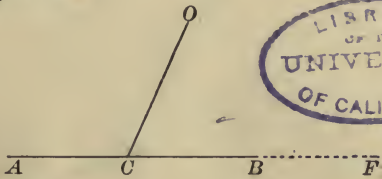
92. COR. Since the angular magnitude about a point is neither increased nor diminished by the number of lines which radiate from the point, it follows that,

The sum of all the angles about a point in a plane is equal to two straight angles, or four right angles.

The sum of all the angles about a point on the same side of a straight line passing through the point, is equal to a straight angle, or two right angles.

Proposition III. Theorem.

93. Conversely: *If two adjacent angles are supplements of each other, their exterior sides lie in the same straight line.*



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Let the adjacent $\angle s$ $OCA + OCB$ = a straight angle.

To prove AC and CB in the same straight line.

Proof. Suppose CF to be in the same line with AC. § 81

Then $\angle OCA + \angle OCF$ is a straight angle. § 90

But $\angle OCA + \angle OCB$ is a straight angle. Hyp.

$\therefore \angle OCA + \angle OCF = \angle OCA + \angle OCB$. Ax. 1

Take away from each of these equals the common $\angle OCA$.

Then $\angle OCF = \angle OCB$. Ax. 3

$\therefore CB$ and CF coincide.

$\therefore AC$ and CB are in the same straight line. Q. E. D.

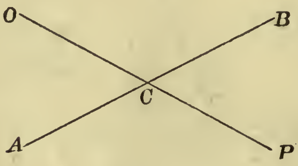
94. Scholium. Since Propositions II. and III. are true, their opposites are true; namely, § 80

If the exterior sides of two adjacent angles are not in a straight line, these angles are not supplements of each other.

If two adjacent angles are not supplements of each other, their exterior sides are not in the same straight line.

PROPOSITION IV. THEOREM.

95. *If one straight line intersects another straight line, the vertical angles are equal.*



Let line OP cut AB at C.

To prove $\angle OCB = \angle ACP$.

Proof. $\angle OCA + \angle OCB = 2$ rt. $\angle$s, § 90

(being sup.-adj. $\angle$s*).*

$\angle OCA + \angle ACP = 2$ rt. $\angle$s, § 90

(being sup.-adj. $\angle$s*).*

$\therefore \angle OCA + \angle OCB = \angle OCA + \angle ACP$. Ax. 1

Take away from each of these equals the common $\angle OCA$.

Then $\angle OCB = \angle ACP$. Ax. 3

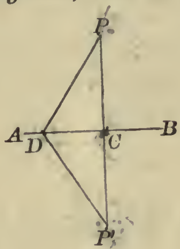
In like manner we may prove

$\angle ACO = \angle PCB$. Q. E. D.

96. COR. *If one of the four angles formed by the intersection of two straight lines is a right angle, the other three angles are right angles.*

Proposition V. Theorem.

97. *From a point without a straight line one perpendicular, and only one, can be drawn to this line.*



Let P be the point and AB the line.

To prove that one perpendicular, and only one, can be drawn from P to AB.

Proof. Turn the part of the plane above AB about AB as an axis until it falls upon the part below AB, and denote by P' the position that P takes.

Turn the revolved plane about AB to its original position, and draw the straight line PP', cutting AB at C.

Take any other point D in AB, and draw PD and $P'D$.

Since PCP' is a straight line, PDP' is not a straight line.
(*Between two points only one straight line can be drawn.*)

$\therefore \angle PCP'$ is a st. $\angle$, and $\angle PDP'$ is not a st. $\angle$.

Turn the figure PCD about AB until P falls upon P'.

Then CP will coincide with CP', and DP with DP'.

$\therefore \angle PCD = \angle P'CD$, and $\angle PDC = \angle P'DC$. § 59

$\therefore \angle PCD$, the half of st. $\angle PCP'$, is a rt. $\angle$; and $\angle PDC$, the half of $\angle PDP'$, is not a rt. $\angle$.

$\therefore PC$ is $\perp$ to AB, and PD is not $\perp$ to AB. § 47

$\therefore$ one $\perp$, and only one, can be drawn from P to AB.

Q. E. D.

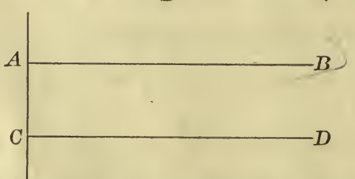
Parallel Lines.

98. Def. *Parallel lines* are lines which lie in the same plane and do not meet however far they are prolonged in both directions.

99. Parallel lines are said to lie in the same direction when they are on the same side of the straight line joining their origins, and in opposite directions when they are on opposite sides of the straight line joining their origins.

Proposition VI.

100. *Two straight lines in the same plane perpendicular to the same straight line are parallel.*



Let AB and CD be perpendicular to AC.

To prove *AB and CD parallel.*

Proof. If AB and CD are not parallel, they will meet if sufficiently prolonged, and we shall have two perpendicular lines from their point of meeting to the same straight line; but this is impossible. § 97

(*From a given point without a straight line, one perpendicular, and only one, can be drawn to the straight line.*)

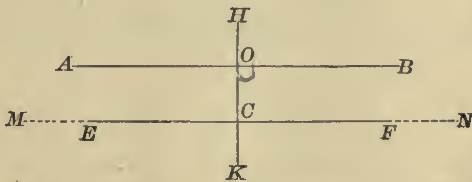
$\therefore$ AB and CD are parallel. Q. E. D

Remark. Here the supposition that AB and CD are *not* parallel leads to the conclusion that two perpendiculars can be drawn from a given point to a straight line. The conclusion is *false*, therefore the supposition is *false;* but if it is false that AB and CD are not parallel, it is true that they are parallel. This method of proof is called the *indirect method.*

101. Ax. *Through a given point, one straight line, and only one, can be drawn parallel to a given straight line.*

Proposition VII. Theorem.

102. *If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other.*



Let AB and EF be two parallel lines, and let HK be perpendicular to AB.

To prove $HK \perp EF$.

Proof. Suppose MN drawn through $C \perp$ to HK.

Then MN is $\parallel$ to AB, § 100

(*two lines in the same plane* $\perp$ *to a given line are parallel*).

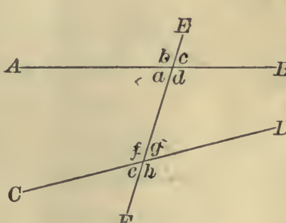
But EF is $\parallel$ to AB. Hyp.

$\therefore EF$ coincides with MN, § 101

(*through the same point only one line can be drawn* $\parallel$ *to a given line*).

$\therefore EF$ is $\perp$ to HK,

that is, HK is $\perp$ to EF. Q. E. D.



103. If two straight lines AB and CD are cut by a third line EF, called a *transversal*, the eight angles formed are named as follows:

The angles a, d, f, g are called *interior*; b, c, e, h are called *exterior* angles.

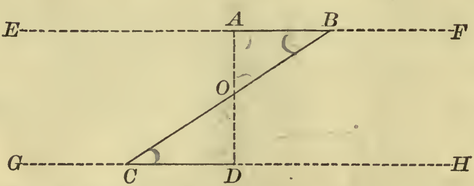
The angles d and f, or a and g, are called *alt.-int.* angles.

The angles b and h, or c and e, are called *alt.-ext.* angles.

The angles f and b, c and g, a and e, or d and h, are called *ext.-int.* angles.

Proposition VIII. Theorem.

104. *If two parallel straight lines are cut by a third straight line, the alternate-interior angles are equal.*



Let EF and GH be two parallel straight lines cut by the line BC.

To prove $\angle B = \angle C$.

Proof. Through O, the middle point of BC, suppose AD drawn $\perp$ to GH.

Then AD is likewise $\perp$ to EF, § 102

(*a straight line $\perp$ to one of two $\parallel$s is $\perp$ to the other*),

that is, CD and BA are both $\perp$ to AD.

Apply figure COD to figure BOA, so that OD shall fall on OA.

Then OC will fall on OB,

(*since $\angle COD = \angle BOA$, being vertical* $\angle$s);

and the point C will fall upon B,

(*since $OC = OB$ by construction*).

Then the $\perp$ CD will coincide with the $\perp$ BA, § 97

(*from a point without a straight line only one $\perp$ to that line can be drawn*).

$\therefore \angle OCD$ coincides with $\angle OBA$, and is equal to it. § 59

Q. E. D.

60

Ex. 1. Find the value of an angle if it is double its complement; if it is one-fourth of its complement. 18

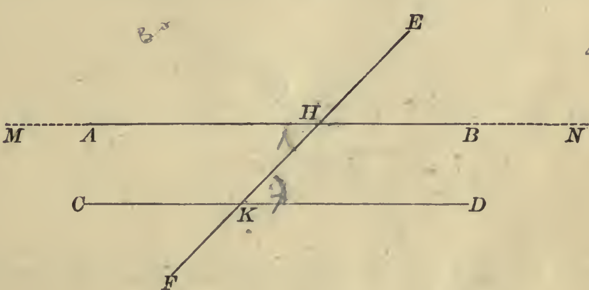
120

Ex. 2. Find the value of an angle if it is double its supplement; if it is one-third of its supplement.

135

PROPOSITION IX. THEOREM.

105. CONVERSELY: *When two straight lines are cut by a third straight line, if the alternate interior angles are equal, the two straight lines are parallel.*



Let EF cut the straight lines AB and CD in the points H and K, and let the $\angle AHK = \angle HKD$.

To prove $AB \parallel$ to CD.

Proof. Suppose MN drawn through $H \parallel$ to CD; § 101

then $\angle MHK = \angle HKD$, § 104

(*being alt.-int.* $\angle$s *of* $\parallel$ *lines*).

But $\angle AHK = \angle HKD$: Hyp.

$\therefore \angle MHK = \angle AHK$. Ax. 1

$\therefore$ the lines MN and AB coincide.

But MN is $\parallel$ to CD. Cons.

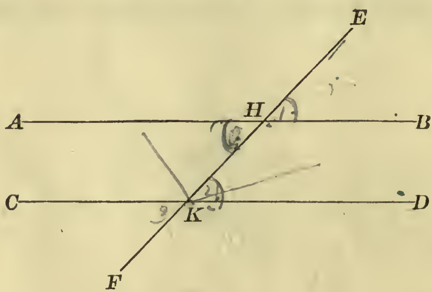
$\therefore AB$, which coincides with MN, is $\parallel$ to CD.

Q.E.D.

Ex. 3. How many degrees in the angle formed by the hands of a clock at 2 o'clock? 3 o'clock? 4 o'clock? 6 o'clock?

Proposition X. Theorem.

106. *If two parallel lines are cut by a third straight line, the exterior-interior angles are equal.*



Let AB and CD be two parallel lines cut by the straight line EF, in the points H and K.

To prove $\angle EHB = \angle HKD$.

Proof. $\angle EHB = \angle AHK$, § 95

(*being vertical* $\angle$s).

But $\angle AHK = \angle HKD$, § 104

(*being alt.-int.* $\angle$s *of* $\parallel$ *lines*).

$\therefore \angle EHB = \angle HKD$. Ax. 1

In like manner we may prove

$$\angle EHA = \angle HKC.$$

Q. E. D.

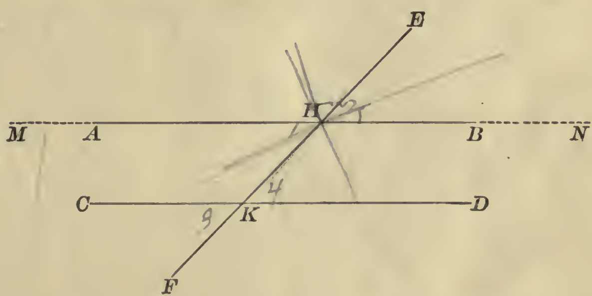
107. Cor. *The alternate-exterior angles EHB and CKF, and also AHE and DKF, are equal.*

Ex. 4. If an angle is bisected, and if a line is drawn through the vertex perpendicular to the bisector, this line forms equal angles with the sides of the given angle.

Ex. 5. If the bisectors of two adjacent angles are perpendicular to each other, the adjacent angles are supplementary.

PROPOSITION XI. THEOREM.

108. CONVERSELY: *When two straight lines are cut by a third straight line, if the exterior-interior angles are equal, these two straight lines are parallel.*



Let EF cut the straight lines AB and CD in the points H and K, and let the $\angle EHB = \angle HKD$.

To prove $AB \parallel$ to CD.

Proof. Suppose MN drawn through $H \parallel$ to CD. § 101

Then $\angle EHN = \angle HKD$, § 106

(*being ext.-int.* $\angle$s *of* $\parallel$ *lines*).

But $\angle EHB = \angle HKD$. Hyp.

$\therefore \angle EHB = \angle EHN$. Ax. 1

$\therefore$ the lines MN and AB coincide.

But MN is $\parallel$ to CD. Cons.

$\therefore AB$, which coincides with MN, is $\parallel$ to CD.

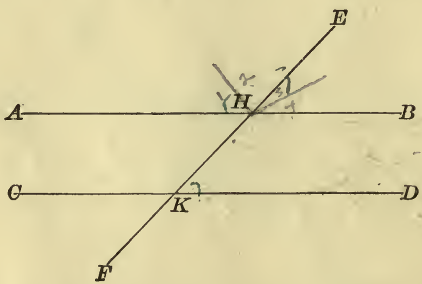
Q. E. D.

Ex. 6. The bisector of one of two vertical angles bisects the other.

Ex. 7. The bisectors of the two pairs of vertical angles formed by two intersecting lines are perpendicular to each other.

Proposition XII. Theorem.

109. ***If two parallel lines are cut by a third straight line, the sum of the two interior angles on the same side of the transversal is equal to two right angles.***



Let AB and CD be two parallel lines cut by the straight line EF in the points H and K.

To prove $\angle BHK + \angle HKD = 2$ *rt.* $\angle$s.

Proof. $\angle EHB + \angle BHK = 2$ rt. $\angle$s, § 90

(*being sup.-adj.* $\angle$s).

But $\angle EHB = \angle HKD$, § 106

(*being ext.-int.* $\angle$s *of* $\parallel$ *lines*).

Substitute $\angle HKD$ for $\angle EHB$ in the first equality;

then $\angle BHK + \angle HKD = 2$ rt. $\angle$s.

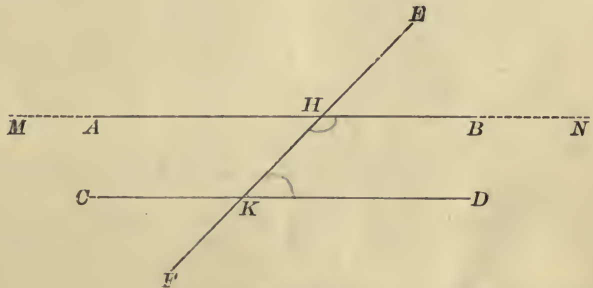
Q. E. D.

Ex. 8. If the angle AHE is an angle of 135°, find the number of degrees in each of the other angles formed at the points H and K.

Ex. 9. Find the angle between the bisectors of adjacent complementary angles.

Proposition XIII. Theorem.

110. Conversely: *When two straight lines are cut by a third straight line, if the two interior angles on the same side of the transversal are together equal to two right angles, then the two straight lines are parallel.*



Let EF cut the straight lines AB and CD in the points H and K, and let the $\angle BHK + \angle HKD$ equal two right angles.

To prove $AB \parallel$ *to* CD.

Proof. Suppose MN drawn through $H \parallel$ to CD.

Then $\angle NHK + \angle HKD = 2$ rt. ∡s, § 109

(*being two interior ∡s of* ∥s *on the same side of the transversal*).

But $\angle BHK + \angle HKD = 2$ rt. ∡s. Hyp.

$\therefore \angle NHK + \angle HKD = \angle BHK + \angle HKD$. Ax. 1

Take away from each of these equals the common $\angle HKD$;

then $\angle NHK = \angle BHK$. Ax. 3

$\therefore$ the lines AB and MN coincide.

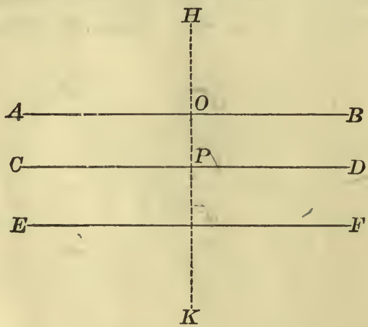
But MN is ∥ to CD. Cons.

$\therefore AB$, which coincides with MN, is ∥ to CD.

Q. E. D.

PROPOSITION XIV. THEOREM.

111. *Two straight lines which are parallel to a third straight line are parallel to each other.*



Let AB and CD be parallel to EF.

To prove $AB \parallel$ *to* CD.

Proof. Suppose HK drawn $\perp$ to EF. § 97

Since CD and EF are $\parallel$, HK is $\perp$ to CD, § 102

(*if a straight line is* $\perp$ *to one of two* $\parallel$*s, it is* $\perp$ *to the other also*).

Since AB and EF are $\parallel$, HK is also $\perp$ to AB. § 102

$$\therefore \angle HOB = \angle HPD,$$

(*each being a rt.* $\angle$).

$\therefore AB$ is $\parallel$ to CD, § 108

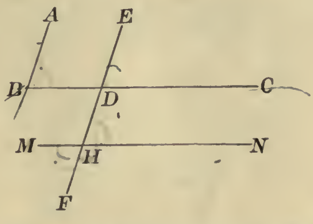
(*when two straight lines are cut by a third straight line, if the ext.-int.* $\angle$s *are equal, the two lines are parallel*).

Q. E. D.

Ex. 10. It has been shown that if two parallels are cut by a transversal, the alternate-interior angles are equal, the exterior-interior angles are equal, the two interior angles on the same side of the transversal are supplementary. State the opposite theorems. State the converse theorems.

Proposition XV. Theorem. /6

112. *Two angles whose sides are parallel each to each, are either equal or supplementary.*



Let AB be parallel to EF, and BC to MN.

To prove $\angle ABC$ *equal to* $\angle EHN$, *and to* $\angle MHF$, *and supplementary to* $\angle EHM$ *and to* $\angle NHF$.

Proof. Prolong (if necessary) BC and FE until they intersect at D. § 81 (2)

Then $\angle B = \angle EDC$, § 106

and $\angle DHN = \angle EDC$.

(*being ext.-int.* $\angle$*s of* $\parallel$ *lines*),

$\therefore \angle B = \angle DHN$; Ax. 1

and $\angle B = \angle MHF$ (the vert. $\angle$ of DHN).

Now $\angle DHN$ is the supplement of $\angle EHM$ and $\angle NHF$.

$\therefore \angle B$, which is equal to $\angle DHN$,

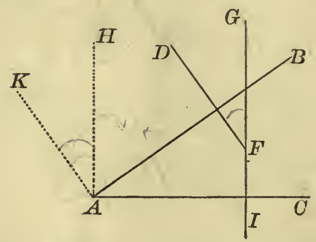
is the supplement of $\angle EHM$ and of $\angle NHF$.

Q. E. D.

Remark. The angles are *equal* when both pairs of parallel sides extend in the same direction, or in opposite directions, from their vertices; the angles are *supplementary* when two of the parallel sides extend in the same direction, and the other two in opposite directions, from their vertices.

Proposition XVI. Theorem.

113. *Two angles whose sides are perpendicular each to each, are either equal or supplementary.*



Let AB be perpendicular to FD, and AC to GI.

To prove $\angle BAC$ *equal to* $\angle DFG$*, and supplementary to* $\angle DFI$.

Proof. Suppose AK drawn $\perp$ to AB, and $AH \perp$ to AC.

Then AK is $\parallel$ to FD, and AH to IG, § 100

(*two lines* $\perp$ *to the same line are parallel*).

$\therefore \angle DFG = \angle KAH$, § 112

(*two angles are equal whose sides are* $\parallel$ *and extend in the same direction from their vertices*).

The $\angle BAK$ is a right angle by construction.

$\therefore \angle BAH$ is the complement of $\angle KAH$.

The $\angle CAH$ is a right angle by construction.

$\therefore \angle BAH$ is the complement of $\angle BAC$.

$\therefore \angle BAC = \angle KAH$, § 87

(*complements of equal angles are equal*).

$\therefore \angle DFG = \angle BAC$. Ax. 1

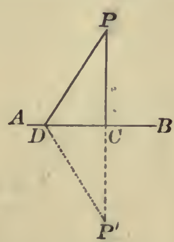
$\therefore \angle DFI$, the supplement of $\angle DFG$, is also the supplement of $\angle BAC$. Q. E. D.

Remark. The angles are *equal* if both are acute or both obtuse; they are *supplementary* if one is acute and the other obtuse.

Perpendicular and Oblique Lines.

Proposition XVII. Theorem.

114. *The perpendicular is the shortest line that can be drawn from a point to a straight line.*



Let AB be the given straight line, P the given point, PC the perpendicular, and PD any other line drawn from P to AB.

To prove $PC < PD$.

Proof. Produce PC to P', making $CP' = PC$; and draw DP'. On AB as an axis, fold over CPD until it comes into the plane of $CP'D$.

The line CP will take the direction of CP',

(since $\angle PCD = \angle P'CD$, each being a rt. $\angle$).

The point P will fall upon the point P',

(since $PC = P'C$ by cons.).

$\therefore$ line $PD =$ line $P'D$,

$\therefore PD + P'D = 2\,PD$,

and $PC + CP' = 2\,PC$. Cons.

But $PC + CP' < PD + DP'$,

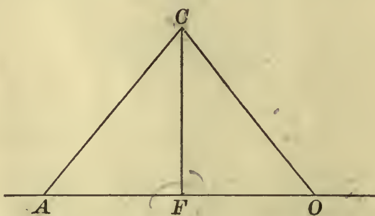
(a straight line is the shortest distance between two points).

$\therefore 2\,PC < 2\,PD$, or $PC < PD$. Q. E. D.

115. Scholium. The *distance* of a point from a line is understood to mean the length of the perpendicular from the point to the line.

Proposition XVIII. Theorem.

116. *Two oblique lines drawn from a point in a perpendicular to a given line, cutting off equal distances from the foot of the perpendicular, are equal.*



Let FC be the perpendicular, and CA and CO two oblique lines cutting off equal distances from F.

To prove $CA = CO$.

Proof. Fold over CFA, on CF as an axis, until it comes into the plane of CFO.

FA will take the direction of FO,

(*since* $\angle CFA = \angle CFO$, *each being a rt.* $\angle$ *by hyp.*).

Point A will fall upon point O,

(*since* $FA = FO$ *by hyp.*).

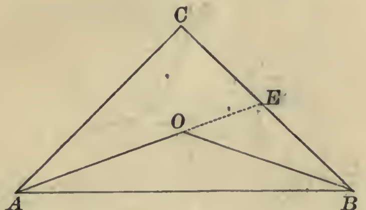
$\therefore$ line $CA =$ line CO,

(*their extremities being the same points*). Q. E. D.

117. Cor. *Two oblique lines drawn from a point in a perpendicular to a given line, cutting off equal distances from the foot of the perpendicular, make equal angles with the given line. and also with the perpendicular.*

PROPOSITION XIX. THEOREM.

118. *The sum of two lines drawn from a point to the extremities of a straight line is greater than the sum of two other lines similarly drawn, but included by them.*



Let CA and CB be two lines drawn from the point C to the extremities of the straight line AB. Let OA and OB be two lines similarly drawn, but included by CA and CB.

To prove $$CA + CB > OA + OB.$$

Proof. Produce AO to meet the line CB at E.

Then $$AC + CE > OA + OE,$$

(*a straight line is the shortest distance between two points*),

and $$BE + OE > BO.$$

Add these inequalities, and we have

$$CA + CE + BE + OE > OA + OE + OB.$$

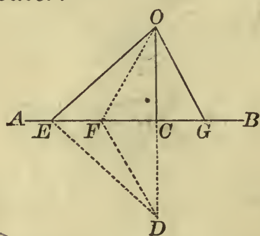
Substitute for $CE + BE$ its equal CB,

and take away OE from each side of the inequality.

We have $$CA + CB > OA + OB. \qquad \text{Ax. 5}$$ Q. E. D.

Proposition XX. Theorem.

119. *Of two oblique lines drawn from the same point in a perpendicular, cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.*



Let OC be perpendicular to AB, OG and OE two oblique lines to AB, and CE greater than CG.

To prove $OE > OG$.

Proof. Take CF equal to CG, and draw OF.

Then $OF = OG$, § 116

(*two oblique lines drawn from a point in a* $\perp$, *cutting off equal distances from the foot of the* $\perp$, *are equal*).

Prolong OC to D, making $CD = OC$.

Draw ED and FD.

Since AB is $\perp$ to OD at its middle point,

$FO = FD$, and $EO = ED$, § 116

But $OE + ED > OF + FD$, § 118

(*the sum of two oblique lines drawn from a point to the extremities of a straight line is greater than the sum of two other lines similarly drawn, but included by them*).

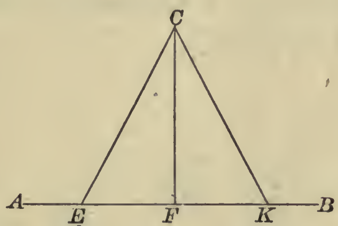
$\therefore 2\,OE > 2\,OF$, or $OE > OF$.

But $OF = OG$. Hence $OE > OG$. Q. E. D.

120. Cor. *Only two equal straight lines can be drawn from a point to a straight line; and of two unequal lines, the greater cuts off the greater distance from the foot of the perpendicular.*

Proposition XXI. Theorem.

121. *Two equal oblique lines, drawn from the same point in a perpendicular, cut off equal distances from the foot of the perpendicular.*



Let CF be the perpendicular, and CE and CK be two equal oblique lines drawn from the point C to AB.

To prove $FE = FK$.

Proof. Fold over CFA on CF as an axis, until it comes into the plane of CFB.

The line FE will take the direction FK,

(*since* $\angle CFE = \angle CFK$, *each being a rt.* $\angle$ *by hyp.*).

Then the point E must fall upon the point K,

and $FE = FK$.

Otherwise one of these oblique lines must be more remote from the perpendicular, and therefore greater than the other; which is contrary to the hypothesis that they are equal. § 119

Q. E. D.

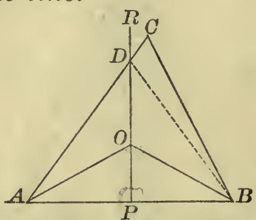
Ex. 11. Show that the bisectors of two supplementary adjacent angles are perpendicular to each other.

Ex. 12. Show that the bisectors of two vertical angles form one straight line.

Ex. 13. Find the complement of an angle containing 26° 52′ 37″. Find the supplement of the same angle.

PROPOSITION XXII. THEOREM.

122. *Every point in the perpendicular, erected at the middle of a given straight line, is equidistant from the extremities of the line, and every point not in the perpendicular is unequally distant from the extremities of the line.*



Let PR be a perpendicular erected at the middle of the straight line AB, O any point in PR, and C any point without PR.

Draw OA and OB, CA and CB.

To prove OA and OB equal, CA and CB unequal.

Proof. $PA = PB$. Hyp.

$\therefore OA = OB$, § 116

(*two oblique lines drawn from the same point in a $\perp$, cutting off equal distances from the foot of the $\perp$, are equal*).

Since C is without the perpendicular, one of the lines, CA or CB, will cut the perpendicular.

Let CA cut the $\perp$ at D, and draw DB.

Then $DB = DA$,

(*two oblique lines drawn from the same point in a $\perp$, cutting off equal distances from the foot of the $\perp$, are equal*).

But $CB < CD + DB$,

(*a straight line is the shortest distance between two points*).

Substitute in this inequality DA for DB, and we have

$$CB < CD + DA.$$

That is, $CB < CA$, Q. E. D.

123. Since two points determine the position of a straight line, *two points equidistant from the extremities of a line determine the perpendicular at the middle of that line.*

THE LOCUS OF A POINT.

124. If it is required to find a point which shall fulfil a *single* geometric condition, the point will have an unlimited number of positions, but will be confined to a *particular line*, or *group of lines.*

Thus, if it is required to find a point equidistant from the extremities of a given straight line, it is obvious from the last proposition that *any* point in the perpendicular to the given line at its middle point does fulfil the condition, and that *no other* point does; that is, the required point is confined to this perpendicular. Again, if it is required to find a point at a given distance from a fixed straight line of indefinite length, it is evident that the point must lie in one of two straight lines, so drawn as to be everywhere at the given distance from the fixed line, one on one side of the fixed line, and the other on the other side.

The *locus of a point* under a given condition is the line, or group of lines, which contains all the points that fulfil the given condition, and no other points.

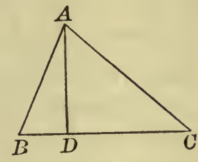
125. SCHOLIUM. In order to prove *completely* that a certain line is the locus of a point under a given condition, it is necessary to prove that *every point in the line satisfies the given condition;* and secondly, that *every point which satisfies the given condition lies in the line* (the converse proposition), or that *every point not in the line does not satisfy the given condition* (the opposite proposition).

126. COR. *The locus of a point equidistant from the extremities of a straight line is the perpendicular bisector of that line.*

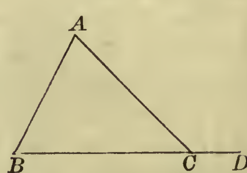
§§ 122, 123

TRIANGLES.

127. A *triangle* is a portion of a plane bounded by three straight lines; as, ABC.



The bounding lines are called the *sides* of the triangle, and their sum is called its *perimeter;* the angles formed by the sides are called the *angles* of the triangle, and the vertices of these angles, the *vertices* of the triangle.



128. An *exterior angle* of a triangle is an angle formed between a side and the prolongation of another side; as, ACD. The interior angle ACB is adjacent to the exterior angle; the other two interior angles, A and B, are called *opposite-interior angles.*


Scalene.

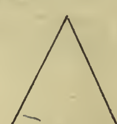

Isosceles.


Equilateral.

129. A triangle is called, with reference to its sides, a *scalene triangle* when no two of its sides are equal; an *isosceles triangle,* when two of its sides are equal; an *equilateral triangle,* when its three sides are equal.


Right.


Obtuse.


Acute.


Equiangular.

130. A triangle is called, with reference to its angles, a *right triangle,* when one of its angles is a right angle; an *obtuse*

triangle, when one of its angles is an obtuse angle; an *acute triangle*, when all three of its angles are acute angles; an *equiangular triangle*, when its three angles are equal.

131. In a right triangle, the side opposite the right angle is called the *hypotenuse*, and the other two sides the *legs*, of the triangle.

132. The side on which a triangle is supposed to stand is called the *base* of the triangle. In the isosceles triangle, the equal sides are called the legs, and the other side, the base; in other triangles, any one of the sides may be taken as the base.

133. The angle opposite the base of a triangle is called the *vertical angle*, and its vertex the *vertex* of the triangle.

134. The *altitude* of a triangle is the perpendicular distance from the vertex to the base, or to the base produced; as, AD.

135. The three perpendiculars from the vertices of a triangle to the opposite sides (produced if necessary) are called the *altitudes;* the three bisectors of the angles are called the *bisectors;* and the three lines from the vertices to the middle points of the opposite sides are called the *medians* of the triangle.

136. If two triangles have the angles of the one equal respectively to the angles of the other, the equal angles are called *homologous angles*, and the sides opposite the equal angles are called *homologous sides.*

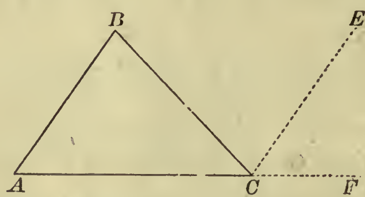
In general, points, lines, and angles, similarly situated in equal or similar figures, are called *homologous.*

137. Theorem. *The sum of two sides of a triangle is greater than the third side, and their difference is less than the third side.*

In the $\triangle ABC$ (Fig. 1), $AB+BC>AC$, for a straight line is the shortest distance between two points; and by taking away BC from both sides, $AB>AC-BC$, or $AC-BC<AB$.

Proposition XXIII. Theorem.

138. *The sum of the three angles of a triangle is equal to two right angles.*



Let ABC be a triangle.

To prove $\angle B + \angle BCA + \angle A = 2$ rt. $\angle$s.

Proof. Suppose CE drawn ∥ to AB, and prolong AC to F.

Then $\angle ECF + \angle ECB + \angle BCA = 2$ rt. $\angle$s, § 92

(*the sum of all the $\angle$s about a point on the same side of a straight line $= 2$ rt. $\angle$s*).

But $\angle A = \angle ECF$, § 106

(*being ext.-int. $\angle$s of ∥ lines*).

and $\angle B = \angle BCE$, § 104

(*being alt.-int. $\angle$s of ∥ lines*).

Substitute for $\angle ECF$ and $\angle BCE$ the equal $\angle$s A and B.

Then $\angle A + \angle B + \angle BCA = 2$ rt. $\angle$s.

Q. E. D.

139. Cor. 1. *If the sum of two angles of a triangle is subtracted from two right angles, the remainder is equal to the third angle.*

140. Cor. 2. *If two triangles have two angles of the one equal to two angles of the other, the third angles are equal.*

141. Cor. 3. *If two right triangles have an acute angle of the one equal to an acute angle of the other, the other acute angles are equal.*

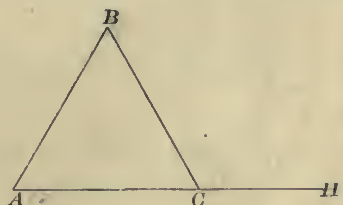
142. Cor. 4. *In a triangle there can be but one right angle, or one obtuse angle.*

143. Cor. 5. *In a right triangle the two acute angles are complements of each other.*

144. Cor. 6. *In an equiangular triangle, each angle is one-third of two right angles, or two-thirds of one right angle.*

Proposition XXIV. Theorem.

145. ***The exterior angle of a triangle is equal to the sum of the two opposite interior angles.***



Let BCH **be an exterior angle of the triangle** ABC.

To prove $\angle BCH = \angle A + \angle B$.

Proof. $\angle BCH + \angle ACB = 2$ rt. $\angle$s,

(*being sup.-adj.* $\angle$s).

$\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s, § 138

(*the sum of the three* $\angle$s *of a* $\triangle = 2$ *rt.* $\angle$s).

$\therefore \angle BCH + \angle ACB = \angle A + \angle B + \angle ACB$. Ax. 1

Take away from each of these equals the common $\angle ACB$;

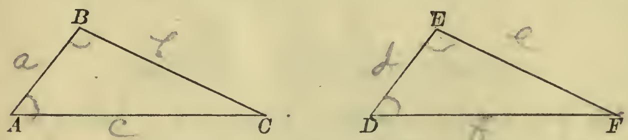
then $\angle BCH = \angle A + \angle B$. Ax. 3

Q. E. D.

146. Cor. *The exterior angle of a triangle is greater than either of the opposite interior angles.*

Proposition XXV. Theorem.

147. *Two triangles are equal if a side and two adjacent angles of the one are equal respectively to a side and two adjacent angles of the other.*



In the triangles ABC ***and*** DEF, ***let*** $AB = DE$, $\angle A = \angle D$, $\angle B = \angle E$.

To prove $\triangle ABC = \triangle DEF$.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ so that AB shall coincide with DE.

AC will take the direction of DF,
(*for* $\angle A = \angle D$, *by hyp.*);

the extremity C of AC will fall upon DF or DF produced.

BC will take the direction of EF,
(*for* $\angle B = \angle E$, *by hyp.*);

the extremity C of BC will fall upon EF or EF produced.

$\therefore$ the point C, falling upon both the lines DF and EF, must fall upon the point common to the two lines, namely, F.

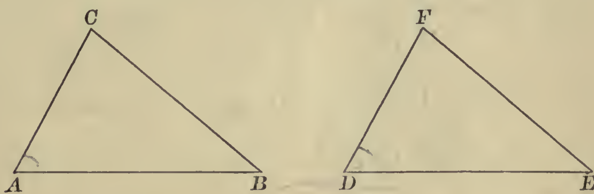
$\therefore$ the two $\triangle$ coincide, and are equal. Q. E. D.

148. Cor. 1. *Two right triangles are equal if the hypotenuse and an acute angle of the one are equal respectively to the hypotenuse and an acute angle of the other.*

149. Cor. 2. *Two right triangles are equal if a side and an acute angle of the one are equal respectively to a side and homologous acute angle of the other.*

Proposition XXVI. Theorem.

150. *Two triangles are equal if two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.*



In the triangles ABC **and** DEF, **let** $AB = DE$, $AC = DF$, $\angle A = \angle D$.

To prove $\triangle ABC = \triangle DEF$.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ so that AB shall coincide with DE.

Then AC will take the direction of DF,

(*for* $\angle A = \angle D$, *by hyp.*);

the point C will fall upon the point F,

(*for* $AC = DF$, *by hyp.*).

$\therefore CB = FE$,

(*their extremities being the same points*).

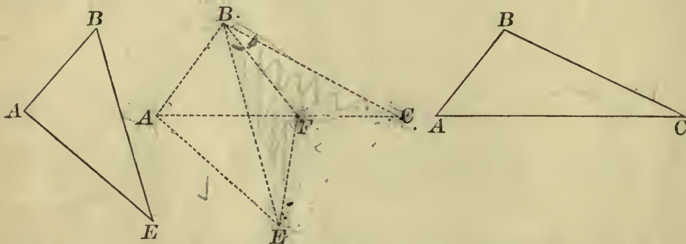
$\therefore$ the two $\triangle$s coincide, and are equal.

Q. E. D.

151. Cor. *Two right triangles are equal if their legs are equal, each to each.*

PROPOSITION XXVII. THEOREM.

152. *If two triangles have two sides of the one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first will be greater than the third side of the second.*



In the triangles ABC and ABE, let $AB = AB$, $BC = BE$; but $\angle ABC$ greater than $\angle ABE$.

To prove $AC > AE$.

Proof. Place the $\triangle$s so that AB of the one shall coincide with AB of the other.

Suppose BF drawn so as to bisect $\angle EBC$.
Draw EF.

In the $\triangle$s EBF and CBF

$EB = BC$, Hyp.

$BF = BF$, Iden.

$\angle EBF = \angle CBF$. Cons.

$\therefore$ the $\triangle$s EBF and CBF are equal, § 150

(*having two sides and the included $\angle$ of one equal respectively to two sides and the included $\angle$ of the other*).

$\therefore EF = FC$,

(*being homologous sides of equal $\triangle$s*).

Now $AF + FE > AE$, § 137

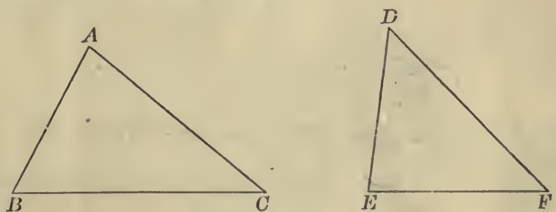
(*the sum of two sides of a $\triangle$ is greater than the third side*).

$\therefore AF + FC > AE$;

or, $AC > AE$. Q. E. D.

PROPOSITION XXVIII. THEOREM.

153. CONVERSELY. *If two sides of a triangle are equal respectively to two sides of another, but the third side of the first triangle is greater than the third side of the second, then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second.*



In the triangles ABC and DEF, let $AB = DE$, $AC = DF$, but let BC be greater than EF.

To prove $\angle A$ *greater than* $\angle D$.

Proof. Now $\angle A$ is equal to $\angle D$, or less than $\angle D$, or greater than $\angle D$.

But $\angle A$ is not equal to $\angle D$, for then $\triangle ABC$ would be equal to $\triangle DEF$, § 150

(*having two sides and the included* $\angle$ *of the one respectively equal to two sides and the included* $\angle$ *of the other*),

and BC would be equal to EF.

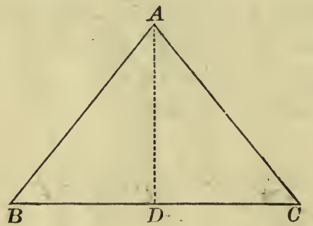
And $\angle A$ is not less than $\angle D$, for then BC would be less than EF. § 152

$\therefore \angle A$ is greater than $\angle D$.

Q. E. D.

Proposition XXIX. Theorem.

154. *In an isosceles triangle the angles opposite the equal sides are equal.*



Let ABC be an isosceles triangle, having the sides AB and AC equal.

To prove $\angle B = \angle C$.

Proof. Suppose AD drawn so as to bisect the $\angle BAC$.

In the $\triangle$ ADB and ADC,

$AB = AC$. Hyp.

$AD = AD$, Iden.

$\angle BAD = \angle CAD$. Cons.

$\therefore \triangle ADB = \triangle ADC$, § 150

(*two* $\triangle$ *are equal if two sides and the included* $\angle$ *of the one are equal respectively to two sides and the included* $\angle$ *of the other*).

$\therefore \angle B = \angle C$. Q. E. D.

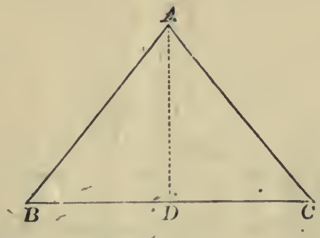
155. Cor. *An equilateral triangle is equiangular, and each angle contains* 60°.

Ex. 14. The bisector of the vertical angle of an isosceles triangle bisects the base, and is perpendicular to the base.

Ex. 15. The perpendicular bisector of the base of an isosceles triangle passes through the vertex and bisects the angle at the vertex.

Proposition XXX. Theorem.

156. *If two angles of a triangle are equal, the sides opposite the equal angles are equal, and the triangle is isosceles.*



In the triangle ABC**, let the** $\angle B = \angle C$.

To prove $AB = AC$.

Proof. Suppose AD drawn $\perp$ to BC.

In the rt. $\triangle$ ADB and ADC,

$AD = AD$, Iden.

$\angle B = \angle C$. Hyp.

$\therefore$ rt. $\triangle ADB =$ rt. $\triangle ADC$, § 149

(*having a side and an acute* $\angle$ *of the one equal respectively to a side and an homologous acute* $\angle$ *of the other*).

$\therefore AB = AC$,

(*being homologous sides of equal* $\triangle$).

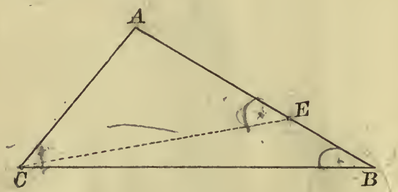
Q. E. D.

157. Cor. *An equiangular triangle is also equilateral.*

Ex. 16. The perpendicular from the vertex to the base of an isosceles triangle is an axis of symmetry.

Proposition XXXI. Theorem.

158. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*



In the triangle ACB let AB be greater than AC.

To prove $\angle ACB$ greater than $\angle B$.

Proof. Take AE equal to AC.

Draw EC.

$\angle AEC = \angle ACE$, § 154

(being $\measuredangle$ opposite equal sides).

But $\angle AEC$ is greater than $\angle B$, § 146

(an exterior $\angle$ of a $\triangle$ is greater than either opposite interior $\angle$).

and $\angle ACB$ is greater than $\angle ACE$. Ax. 8

Substitute for $\angle ACE$ its equal $\angle AEC$,

then $\angle ACB$ is greater than $\angle AEC$.

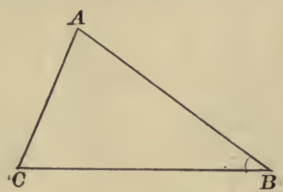
Much more, then, is the $\angle ACB$ greater than $\angle B$.

Q. E. D.

Ex. 17. If the angles ABC and ACB, at the base of an isosceles triangle, be bisected by the straight lines BD, CD, show that DBC will be an isosceles triangle.

PROPOSITION XXXII. THEOREM.

159. CONVERSELY: *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle.*



In the triangle ACB, let angle ACB be greater than angle B.

To prove $AB > AC$.

Proof. Now AB is equal to AC, or less than AC, or greater than AC.

But AB is not equal to AC, for then the $\angle C$ would be equal to the $\angle B$, § 154

(being ∠s opposite equal sides).

And AB is not less than AC, for then the $\angle C$ would be less than the $\angle B$, § 158

(if two sides of a △ are unequal, the ∠s opposite are unequal, and the greater ∠ is opposite the greater side).

$\therefore$ AB is greater than AC.

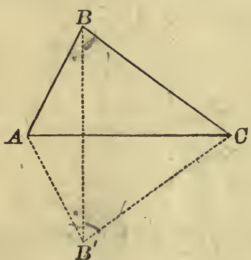
Q. E. D.

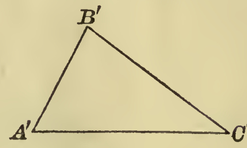
Ex. 18. ABC and ABD are two triangles on the same base AB, and on the same side of it, the vertex of each triangle being without the other. If AC equal AD, show that BC cannot equal BD.

Ex. 19. The sum of the lines which join a point within a triangle to the three vertices is less than the perimeter, but greater than half the perimeter.

Proposition XXXIII. Theorem.

160. *Two triangles are equal if the three sides of the one are equal respectively to the three sides of the other.*





In the triangles ABC *and* $A'B'C'$, *let* $AB = A'B'$, $AC = A'C'$, $BC = B'C'$.

To prove $\triangle ABC = \triangle A'B'C'$.

Proof. Place $\triangle A'B'C'$ in the position $AB'C$, having its greatest side $A'C'$ in coincidence with its equal AC, and its vertex at B', opposite B; and draw BB'.

Since $AB = AB'$, Hyp.

$\angle ABB' = \angle AB'B$, § 154

(*in an isosceles* $\triangle$ *the* $\angle$s *opposite the equal sides are equal*).

Since $CB = CB'$, Hyp.

$\angle CBB' = \angle CB'B$. § 154

Hence, $\angle ABC = \angle AB'C$, Ax. 2

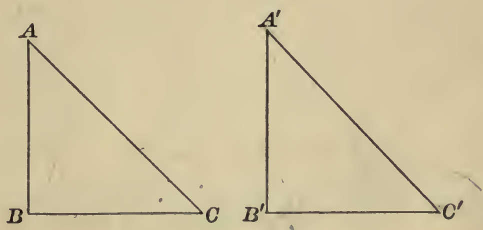
$\therefore \triangle ABC = \triangle AB'C = \triangle A'B'C'$ § 150

(*two* $\triangle$s *are equal if two sides and included* $\angle$ *of one are equal to two sides and included* $\angle$ *of the other*).

Q. E. D.

Proposition XXXIV. Theorem.

161. *Two right triangles are equal if a side and the hypotenuse of the one are equal respectively to a side and the hypotenuse of the other.*



In the right triangles ABC *and* $A'B'C'$, *let* $AB = A'B'$, *and* $AC = A'C'$.

To prove $\triangle ABC = \triangle A'B'C'$.

Proof. Apply the $\triangle ABC$ to the $\triangle A'B'C'$, so that AB shall coincide with $A'B'$, A falling upon A', B upon B', and C and C' upon the same side of $A'B'$.

Then BC will take the direction of $B'C'$,

(*for* $\angle ABC = \angle A'B'C'$, *each being a rt.* $\angle$).

Since $AC = A'C'$,

the point C will fall upon C', § 121

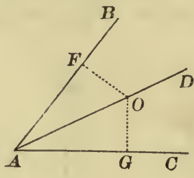
(*two equal oblique lines from a point in a* $\perp$ *cut off equal distances from the foot of the* $\perp$).

$\therefore$ the two $\triangle$s coincide, and are equal.

Q. E. D.

PROPOSITION XXXV. THEOREM.

162. *Every point in the bisector of an angle is equidistant from the sides of the angle.*



Let AD be the bisector of the angle BAC, and let O be any point in AD.

To prove that O is equidistant from AB and AC.

Proof. Draw OF and $OG \perp$ to AB and AC respectively.

In the rt. $\triangle$ AOF and AOG

$$AO = AO, \qquad \text{Iden.}$$

$$\angle BAO = \angle CAO. \qquad \text{Hyp.}$$

$$\therefore \triangle AOF = \triangle AOG, \qquad \S\ 148$$

(*two rt.* $\triangle$ *are equal if the hypotenuse and an acute* $\angle$ *of the one are equal respectively to the hypotenuse and an acute* $\angle$ *of the other*).

$$\therefore OF = OG,$$

(*homologous sides of equal* $\triangle$).

$$\therefore O \text{ is equidistant from } AB \text{ and } AC.$$

Q. E. D.

What is the locus of a point:

Ex. 20. At a given distance from a fixed point? § 57.

Ex. 21. Equidistant from two fixed points? § 119.

Ex. 22. At a given distance from a fixed straight line of indefinite length?

Ex. 23. Equidistant from two given parallel lines?

Ex. 24. Equidistant from the extremities of a given line?

PROPOSITION XXXVI. THEOREM.

163. *Every point within an angle, and equidistant from its sides, is in the bisector of the angle.*

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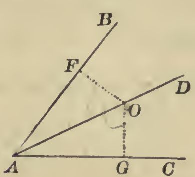


Let O be equidistant from the sides of the angle BAC, and let AO join the vertex A and the point O.

To prove that AO is the bisector of $\angle BAC$.

Proof. Suppose OF and OG drawn $\perp$ to AB and AC, respectively.

In the rt. $\triangle$ AOF and AOG

$$OF = OG, \qquad \text{Hyp.}$$

$$AO = AO. \qquad \text{Iden.}$$

$$\therefore \triangle AOF = \triangle AOG, \qquad \S\, 161$$

(*two rt.* $\triangle$ *are equal if the hypotenuse and a side of the one are equal to the hypotenuse and a side of the other*).

$$\therefore \angle FAO = \angle GAO,$$

(*homologous* $\angle$ *of equal* $\triangle$).

$$\therefore AO \text{ is the bisector of } \angle BAC.$$

Q. E. D.

164. COR. *The locus of a point within an angle, and equidistant from its sides, is the bisector of the angle.*

QUADRILATERALS.

165. A *quadrilateral* is a portion of a plane bounded by four straight lines.

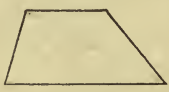
The bounding lines are the *sides*, the angles formed by these sides are the *angles*, and the vertices of these angles are the *vertices*, of the quadrilateral.

166. A *trapezium* is a quadrilateral which has no two sides parallel.

167. A *trapezoid* is a quadrilateral which has two sides, and only two sides, parallel.

168. A *parallelogram* is a quadrilateral which has its opposite sides parallel.

Trapezium.



Trapezoid.

Parallelogram.

169. A *rectangle* is a parallelogram which has its angles right angles.

170. A *rhomboid* is a parallelogram which has its angles oblique angles.

171. A *square* is a rectangle which has its sides equal.

172. A *rhombus* is a rhomboid which has its sides equal.

Square.

Rectangle.

Rhombus.

Rhomboid.

173. The side upon which a parallelogram stands, and the opposite side, are called its lower and upper *bases*.

174. The parallel sides of a trapezoid are called its *bases*, the other two sides its *legs*, and the line joining the middle points of the legs is called the *median*.

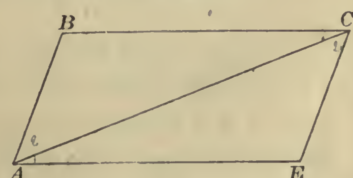
175. A trapezoid is called an *isosceles trapezoid* when its legs are equal.

176. The *altitude* of a parallelogram or trapezoid is the perpendicular distance between its bases.

177. The *diagonal* of a quadrilateral is a straight line joining two opposite vertices.

Proposition XXXVII. Theorem.

178. *The diagonal of a parallelogram divides the figure into two equal triangles.*



Let $ABCE$ be a parallelogram and AC its diagonal.

To prove $\triangle ABC = \triangle AEC$.

In the $\triangle$s ABC and AEC,

$AC = AC$, Iden.

$\angle ACB = \angle CAE$, § 104

and $\angle CAB = \angle ACE$,

(*being alt.-int.* $\angle$*s of* $\parallel$ *lines.*)

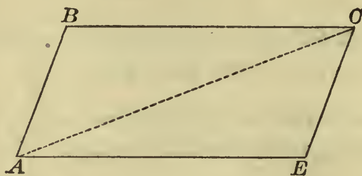
$\therefore \triangle ABC = \triangle AEC$, § 147

(*having a side and two adj.* $\angle$*s of the one equal respectively to a side and two adj.* $\angle$*s of the other.*)

Q. E. D.

Proposition XXXVIII. Theorem.

179. *In a parallelogram the opposite sides are equal, and the opposite angles are equal.*



Let the figure ABCE be a parallelogram.

To prove $BC = AE$, and $AB = EC$,

also, $\angle B = \angle E$, and $\angle BAE = \angle BCE$.

Proof. Draw AC.

$\triangle ABC = \triangle AEC$, § 178

(*the diagonal of a* ▱ *divides the figure into two equal* ▵s).

$\therefore BC = AE$, and $AB = CE$,

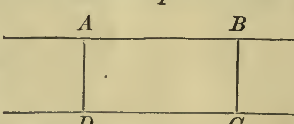
(*being homologous sides of equal* ▵s).

Also, $\angle B = \angle E$, and $\angle BAE = \angle BCE$, § 112

(*having their sides* ∥ *and extending in opposite directions from their vertices*).

Q. E. D.

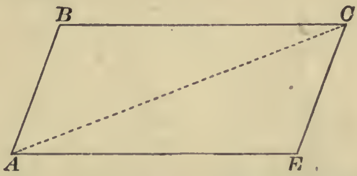
180. Cor. *Parallel lines comprehended between parallel lines are equal.*



181. Cor. 2. *Two parallel lines are everywhere equally distant.* For if AB and DC are parallel, ⊥s dropped from *any* points in AB to DC, measure the distances of these points from DC. But these ⊥s are equal, by § 180; hence, *all* points in AB are equidistant from DC.

Proposition XXXIX. Theorem.

182. *If two sides of a quadrilateral are equal and parallel, then the other two sides are equal and parallel, and the figure is a parallelogram.*



Let the figure $ABCE$ be a quadrilateral, having the side AE equal and parallel to BC.

To prove AB *equal and* $\parallel$ *to* EC.

Proof. Draw AC.

In the $\triangle$ ABC and AEC

$BC = AE$, Hyp.

$AC = AC$, Iden.

$\angle BCA = \angle CAE$, § 104

(being alt.-int. $\angle$s of $\parallel$ lines).

$\therefore \triangle ABC = \triangle ACE$, § 150

(having two sides and the included $\angle$ of the one equal respectively to two sides and the included $\angle$ of the other).

$\therefore AB = EC$,

(being homologous sides of equal $\triangle$s).

Also, $\angle BAC = \angle ACE$,

(being homologous $\angle$s of equal $\triangle$s).

$\therefore AB$ is $\parallel$ to EC, § 105

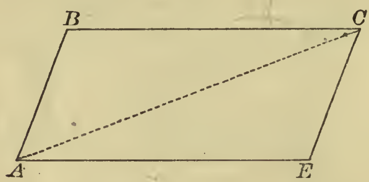
(when two straight lines are cut by a third straight line, if the alt.-int. $\angle$s are equal, the lines are parallel).

$\therefore$ the figure $ABCE$ is a $\square$, § 168

(the opposite sides being parallel). Q. E. D.

PROPOSITION XL. THEOREM.

183. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Let the figure $ABCE$ be a quadrilateral having $BC = AE$ and $AB = EC$.

To prove figure $ABCE$ a ▱.

Proof. Draw AC.

In the ⊿ ABC and AEC

$BC = AE$, Hyp.

$AB = CE$, Hyp.

$AC = AC$. Iden.

$\therefore \triangle ABC = \triangle AEC$, § 160

(*having three sides of the one equal respectively to three sides of the other*).

$\therefore \angle ACB = \angle CAE$,

and $\angle BAC = \angle ACE$,

(*being homologous* ∠ *of equal* ⊿).

$\therefore BC$ is ∥ to AE,

and AB is ∥ to EC, § 105

(*when two straight lines lying in the same plane are cut by a third straight line, if the alt.-int.* ∠ *are equal, the lines are parallel*).

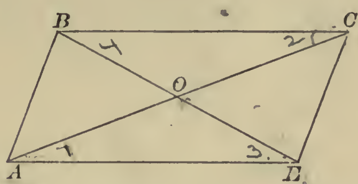
$\therefore$ the figure $ABCE$ is a ▱, § 168

(*having its opposite sides parallel*).

Q. E. D.

Proposition XLI. Theorem.

184. *The diagonals of a parallelogram bisect each other.*



Let the figure $ABCE$ be a parallelogram, and let the diagonals AC and BE cut each other at O.

To prove $AO = OC$, and $BO = OE$.

In the △ AOE and BOC

$$AE = BC, \qquad \S\,179$$

(*being opposite sides of a* ▱).

$$\angle OAE = \angle OCB, \qquad \S\,104$$

and $\angle OEA = \angle OBC$,

(*being alt.-int. ∠ of ∥ lines*).

$$\therefore \triangle AOE = \triangle BOC, \qquad \S\,147$$

(*having a side and two adj. ∠ of the one equal respectively to a side and two adj. ∠ of the other*).

$$\therefore AO = OC, \text{ and } BO = OE,$$

(*being homologous sides of equal △*).

Q. E. D.

Ex. 25. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

Ex. 26. The diagonals of a rectangle are equal.

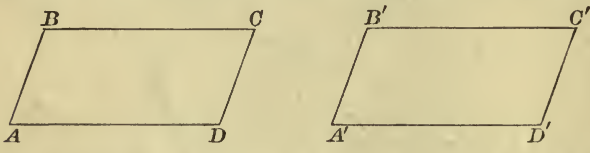
Ex. 27. If the diagonals of a parallelogram are equal, the figure is a rectangle.

Ex. 28. The diagonals of a rhombus are perpendicular to each other, and bisect the angles of the rhombus.

Ex. 29. The diagonals of a square are perpendicular to each other, and bisect the angles of the square.

Proposition XLII. Theorem.

185. *Two parallelograms, having two sides and the included angle of the one equal respectively to two sides and the included angle of the other, are equal.*



In the parallelograms ABCD and A'B'C'D', let $AB = A'B'$, $AD = A'D'$, *and* $\angle A = \angle A'$.

To prove that the ▱s *are equal.*

Apply ▱ $ABCD$ to ▱ $A'B'C'D'$, so that AD will fall on and coincide with $A'D'$.

Then AB will fall on $A'B'$,
(for $\angle A = \angle A'$, *by hyp.)*,
and the point B will fall on B',
(for $AB = A'B'$, *by hyp.)*.

Now, BC and $B'C'$ are both ∥ to $A'D'$ and are drawn through point B'.

∴ the lines BC and $B'C'$ coincide, § 101

and C falls on $B'C'$ or $B'C'$ produced.

In like manner, DC and $D'C'$ are ∥ to $A'B'$ and are drawn through the point D'.

∴ DC and $D'C'$ coincide. § 101

∴ the point C falls on $D'C'$, or $D'C'$ produced.

∴ C falls on both $B'C'$ and $D'C'$.

∴ C must fall on the point common to both, namely, C'.

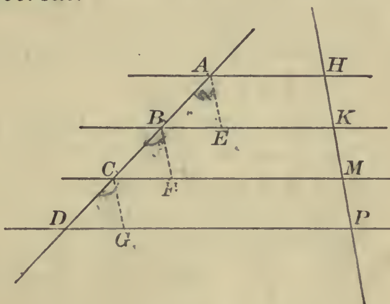
∴ the two ▱s coincide, and are equal.

Q. E. D

186. Cor. *Two rectangles having equal bases and altitudes are equal.*

Proposition XLIII. Theorem.

187. *If three or more parallels intercept equal parts on any transversal, they intercept equal parts on every transversal.*



Let the parallels AH, BK, CM, DP intercept equal parts HK, KM, MP on the transversal HP.

To prove that they intercept equal parts AB, BC, CD on the transversal AD.

Proof. From A, B, and C suppose AE, BF, and CG drawn $\parallel$ to HP.

Then $AE = HK$, $BF = KM$, $CG = MP$, § 180
(*parallels comprehended between parallels are equal*).

$\therefore AE = BF = CG$. Ax. 1

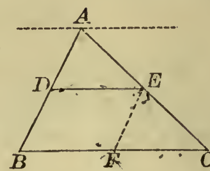
Also $\angle BAE = \angle CBF = \angle DCG$, § 106
(*being ext.-int. $\angle$s of $\parallel$ lines*);

and $\angle AEB = \angle BFC = \angle CGD$, § 112
(*having their sides $\parallel$ and directed the same way from the vertices*).

$\therefore \triangle ABE = \triangle BCF = \triangle CDG$, § 147
(*each having a side and two adj. $\angle$s respectively equal to a side and two adj. $\angle$s of the others*).

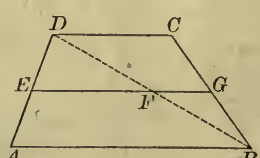
$\therefore AB = BC = CD$,
(*homologous sides of equal $\triangle$s*). Q. E. D.

188. Cor. 1. *The line parallel to the base of a triangle and bisecting one side, bisects the other side also.* For, let DE be $\parallel$ to BC and bisect AB. Draw through A a line $\parallel$ to BC. Then this line is $\parallel$ to DE, by § 111. The three parallels by hypothesis intercept equal parts on the transversal AB, and therefore, by § 187, they intercept equal parts on the transversal AC; that is, the line DE bisects AC.



189. Cor. 2. *The line which joins the middle points of two sides of a triangle is parallel to the third side, and is equal to half the third side.* For, a line drawn through D, the middle point of AB, $\parallel$ to BC, passes through E, the middle point of AC, by § 188. Therefore, the line joining D and E coincides with this parallel and is $\parallel$ to BC. Also, since EF drawn $\parallel$ to AB bisects AC, it bisects BC, by § 188; that is, $BF = FC = \frac{1}{2} BC$. But $BDEF$ is a $\square$ by construction, and therefore $DE = BF = \frac{1}{2} BC$.

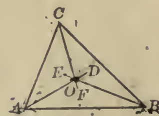
190. Cor. 3. *The line which is parallel to the bases of a trapezoid and bisects one leg of the trapezoid bisects the other leg also.* For if parallels intercept equal parts on any transversal, they intercept equal parts on every transversal by § 187.



191. Cor. 4. *The median of a trapezoid is parallel to the bases, and is equal to half the sum of the bases.* For, draw the diagonal DB. In the $\triangle\, ADB$ join E, the middle point of AD, to F, the middle point of DB. Then, by § 189, EF is $\parallel$ to AB and $= \frac{1}{2} AB$. In the $\triangle\, DBC$ join F to G, the middle point of BC. Then FG is $\parallel$ to DC and $= \frac{1}{2} DC$. AB and FG, being $\parallel$ to DC, are $\parallel$ to each other. But only one line can be drawn through F $\parallel$ to AB. Therefore FG is the prolongation of EF. Hence EFG is $\parallel$ to AB and DC, and $= \frac{1}{2}(AB + DC)$.

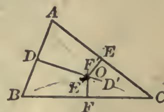
Exercises.

30. The bisectors of the angles of a triangle meet in a point which is equidistant from the sides of the triangle.



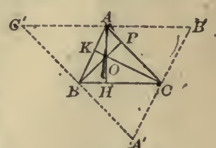
HINT. Let the bisectors AD and BE intersect at O. Then O being in AD is equidistant from AC and AB. (Why?) And O being in BE is equidistant from BC and AB. Hence O is equidistant from AC and BC, and therefore is in the bisector CF. (Why?)

31. The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the vertices of the triangle.



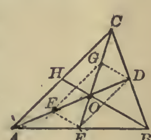
HINT. Let the ⊥ bisectors EE' and DD' intersect at O. Then O being in EE' is equidistant from A and C. (Why?) And O being in DD' is equidistant from A and B. Hence O is equidistant from B and C, and therefore is in the ⊥ bisector FF'. (Why?)

32. The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.



HINT. Let the ⊥s be AH, BP, and CK. Through A, B, C suppose $B'C'$, $A'C'$, $A'B'$ drawn ∥ to BC, AC, AB, respectively. Then AH is ⊥ to $B'C'$. (Why?) Now $ABCB'$ and $ACBC'$ are ▱s (why?), and $AB' = BC$, and $AC' = BC$. (Why?) That is, A is the middle point of $B'C'$. In the same way, B and C are the middle points of $A'C'$ and $A'B'$, respectively. Therefore, AH, BP, and CK are the ⊥ bisectors of the sides of the △ $A'B'C'$. Hence they meet in a point. (Why?)

33. The medians of a triangle meet in a point which is two-thirds of the distance from each vertex to the middle of the opposite side.



HINT. Let the two medians AD and CE meet in O. Take F the middle point of OA, and G of OC. Join GF, FE, ED, and DG. In △ AOC, GF is ∥ to AC and equal to $\frac{1}{2} AC$. (Why?) DE is ∥ to AC and equal to $\frac{1}{2} AC$. (Why?) Hence $DGFE$ is a ▱. (Why?) Hence $AF = FO = OD$, and $CG = GO = OE$. (Why?) Hence, *any median* cuts off *on any other median* two-thirds of the distance from the vertex to the middle of the opposite side. Therefore the median from B will cut off AO, two-thirds of AD; that is, will pass through O.

Polygons in General.

192. A *polygon* is a plane figure bounded by straight lines.

The bounding lines are the *sides* of the polygon, and their sum is the *perimeter* of the polygon.

The *angles* which the adjacent sides make with each other are the angles of the polygon, and their vertices are the vertices of the polygon.

The number of sides of a polygon is evidently equal to the number of its angles.

193. A *diagonal* of a polygon is a line joining the vertices of two angles not adjacent; as AC, Fig. 1.

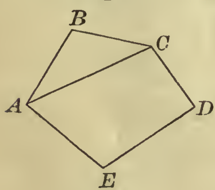


Fig. 1.

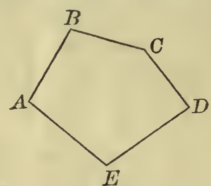


Fig. 2.

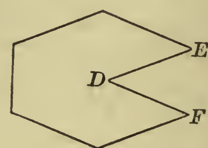


Fig. 3.

194. An *equilateral* polygon is a polygon which has all its sides equal.

195. An *equiangular* polygon is a polygon which has all its angles equal.

196. A *convex* polygon is a polygon of which no side, when produced, will enter the surface bounded by the perimeter.

197. Each angle of such a polygon is called a *salient* angle, and is less than a straight angle.

198. A *concave* polygon is a polygon of which two or more sides, when produced, will enter the surface bounded by the perimeter. Fig. 3.

199. The angle FDE is called a *re-entrant* angle, and is greater than a straight angle.

If the term polygon is used, a *convex* polygon is meant.

200. Two polygons are *equal* when they can be divided by diagonals into the same number of triangles, equal each to each, and similarly placed; for the polygons can be applied to each other, and the corresponding triangles will evidently coincide.

201. Two polygons are *mutually equiangular*, if the angles of the one are equal to the angles of the other, each to each, when taken in the same order. Figs. 1 and 2.

202. The equal angles in mutually equiangular polygons are called *homologous* angles; and the sides which lie *between* equal angles are called *homologous* sides.

203. Two polygons are *mutually equilateral*, if the sides of the one are equal to the sides of the other, each to each, when taken in the same order. Figs. 1 and 2.

FIG. 4.

FIG. 5.

FIG. 6.

FIG. 7.

Two polygons may be mutually equiangular without being mutually equilateral; as, Figs. 4 and 5.

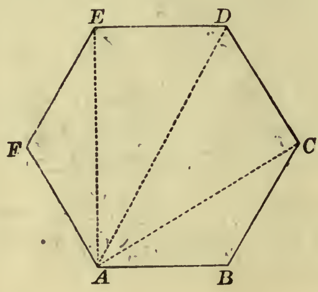
And, *except in the case of triangles*, two polygons may be mutually equilateral without being mutually equiangular; as, Figs. 6 and 7.

If two polygons are mutually equilateral and equiangular, *they are equal*, for they may be applied the one to the other so as to coincide.

204. A polygon of three sides is called a *trigon* or *triangle;* one of four sides, a *tetragon* or *quadrilateral;* one of five sides, a *pentagon;* one of six sides, a *hexagon;* one of seven sides, a *heptagon;* one of eight sides, an *octagon;* one of ten sides, a *decagon;* one of twelve sides, a *dodecagon.*

Proposition XLIV. Theorem.

205. *The sum of the interior angles of a polygon is equal to two right angles, taken as many times less two as the figure has sides.*



Let the figure ABCDEF be a polygon having n sides.

To prove $\angle A + \angle B + \angle C$, etc. $= (n-2)\,2$ *rt.* $\angle$s.

Proof. From the vertex A draw the diagonals AC, AD, and AE.

The sum of the $\angle$s of the $\triangle$s = the sum of the $\angle$s of the polygon.

Now there are $(n-2)$ $\triangle$s,

and the sum of the $\angle$s of each $\triangle = 2$ rt. $\angle$s. § 138

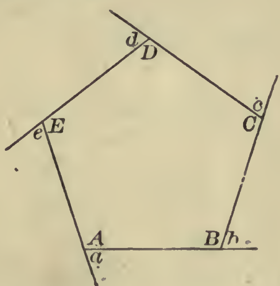
$\therefore$ the sum of the $\angle$s of the $\triangle$s, that is, the sum of the $\angle$s of the polygon $= (n-2)\,2$ rt. $\angle$s.

Q. E. D.

206. Cor. *The sum of the angles of a quadrilateral equals two right angles taken* $(4-2)$ *times*, i.e., *equals* 4 *right angles; and if the angles are all equal, each angle is a right angle. In general, each angle of an equiangular polygon of* n *sides is equal to* $\frac{2(n-2)}{n}$ *right angles.*

Proposition XLV. Theorem.

207. *The exterior angles of a polygon, made by producing each of its sides in succession, are together equal to four right angles.*



Let the figure ABCDE be a polygon, having its sides produced in succession.

To prove the sum of the ext. $\measuredangle s = 4$ *rt.* $\measuredangle s$.

Proof. Denote the int. $\measuredangle s$ of the polygon by A, B, C, D, E, and the ext. $\measuredangle s$ by a, b, c, d, e.

$$\angle A + \angle a = 2 \text{ rt. } \measuredangle s, \qquad \S\, 90$$

and $$\angle B + \angle b = 2 \text{ rt. } \measuredangle s,$$

(being sup.-adj. $\measuredangle s$*).*

In like manner each pair of adj. $\measuredangle s = 2$ rt. $\measuredangle s$.

$\therefore$ the sum of the interior and exterior $\measuredangle s = 2$ rt. $\measuredangle s$ taken as many times as the figure has sides,

or, $$2n \text{ rt. } \measuredangle s.$$

But the interior $\measuredangle s = 2$ rt. $\measuredangle s$ taken as many times as the figure has sides less two, $= (n - 2)\, 2$ rt. $\measuredangle s$,

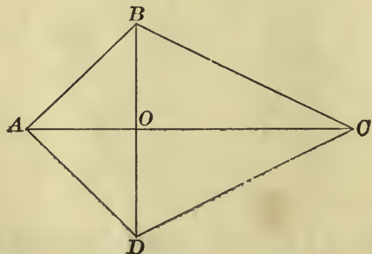
or, $$2n \text{ rt. } \measuredangle s - 4 \text{ rt. } \measuredangle s.$$

$$\therefore \text{ the exterior } \measuredangle s = 4 \text{ rt. } \measuredangle s.$$

Q. E. D.

Proposition XLVI. Theorem.

208. *A quadrilateral which has two adjacent sides equal, and the other two sides equal, is symmetrical with respect to the diagonal joining the vertices of the angles formed by the equal sides, and the diagonals intersect at right angles.*



Let $ABCD$ be a quadrilateral, having $AB = AD$, and $CB = CD$, and having the diagonals AC and BD.

To prove that the diagonal AC is an axis of symmetry, and is $\perp$ to the diagonal BD.

Proof. In the $\triangle$ ABC and ADC

$$AB = AD, \text{ and } BC = DC, \qquad \text{Hyp.}$$

and $$AC = AC. \qquad \text{Iden.}$$

$$\therefore \triangle ABC = \triangle ADC, \qquad \S\ 160$$

(*having three sides of the one equal to three sides of the other*).

$$\therefore \angle BAC = \angle DAC, \text{ and } \angle BCA = \angle DCA,$$

(*homologous* $\angle$ *of equal* $\triangle$).

Hence, if ABC is turned on AC as an axis, AB will fall upon AD, CB on CD, and OB on OD.

Hence AC is an axis of symmetry, § 65, and is $\perp$ to BD.

Q. E. D.

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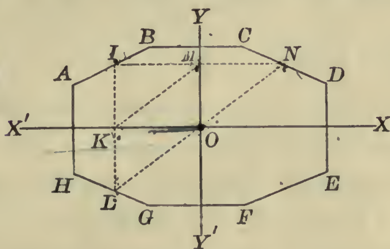
PROPOSITION XLVII. THEOREM.

209. *If a figure is symmetrical with respect to two axes perpendicular to each other, it is symmetrical with respect to their intersection as a centre.*



Let the figure $ABCDEFGH$ be symmetrical with respect to the two axes XX', YY', which intersect at O.

To prove O the centre of symmetry of the figure.

Proof. Let N be any point in the perimeter of the figure.

Draw $NMI \perp$ to YY', and $IKL \perp$ to XX'.

Join LO, ON, and KM.

Now $KI = KL$, § 61

(*the figure being symmetrical with respect to XX'*).

But $KI = OM$, § 180

(∥s *comprehended between* ∥s *are equal*).

$\therefore KL = OM$, and $KLOM$ is a ▱, § 182

(*having two sides equal and parallel*).

$\therefore LO$ is equal and parallel to KM. § 179

In like manner we may prove ON equal and parallel to KM. Hence the points L, O, and N are in the same straight line drawn through the point O ∥ to KM; and $LO = ON$, since each is equal to KM.

$\therefore$ any straight line LON, drawn through O, is bisected at O.

$\therefore O$ is the centre of symmetry of the figure. § 64

Q. E. D.

Exercises.

34. The median from the vertex to the base of an isosceles triangle is perpendicular to the base, and bisects the vertical angle.

35. State and prove the converse.

36. The bisector of an exterior angle of an isosceles triangle, formed by producing one of the legs through the vertex, is parallel to the base.

37. State and prove the converse.

38. The altitudes upon the legs of an isosceles triangle are equal.

39. State and prove the converse.

40. The medians drawn to the legs of an isosceles triangle are equal.

41. State and prove the converse. (See Ex. 33.)

42. The bisectors of the base angles of an isosceles triangle are equal.

43. State the converse and the opposite theorems.

44. The perpendiculars dropped from the middle point of the base of an isosceles triangle upon the legs are equal.

45. State and prove the converse.

46. If one of the legs of an isosceles triangle is produced through the vertex by its own length, the line joining the end of the leg produced to the nearer end of the base is perpendicular to the base.

47. Show that the sum of the interior angles of a hexagon is equal to eight right angles.

48. Show that each angle of an equiangular pentagon is $\frac{6}{5}$ of a right angle.

49. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

50. How many sides has a polygon, the sum of whose interior angles is equal to the sum of its exterior angles?

51. How many sides has a polygon, the sum of whose interior angles is double that of its exterior angles?

52. How many sides has a polygon, the sum of whose exterior angles is double that of its interior angles?

53. BAC is a triangle having the angle B double the angle A. If BD bisect the angle B, and meet AC in D, show that BD is equal to AD.

54. If from any point in the base of an isosceles triangle parallels to the legs are drawn, show that a parallelogram is formed whose perimeter is constant, and equal to the sum of the legs of the triangle.

55. The lines joining the middle points of the sides of a triangle divide the triangle into four equal triangles.

56. The lines joining the middle points of the side of a square, taken in order, enclose a square.

57. The lines joining the middle points of the sides of a rectangle (not a square), taken in order, enclose a rhombus.

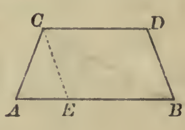
58. The lines joining the middle points of the sides of a rhombus, taken in order, enclose a rectangle.

59. The lines joining the middle points of the sides of an isosceles trapezoid, taken in order, enclose a rhombus or a square.

60. The lines joining the middle points of the sides of any quadrilateral, taken in order, enclose a parallelogram.

61. The median of a trapezoid passes through the middle points of the two diagonals.

62. The line joining the middle points of the diagonals of a trapezoid is equal to half the difference of the bases.



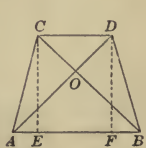
63. In an isosceles trapezoid each base makes equal angles with the legs.

HINT. Draw $CE \parallel DB$.

64. In an isosceles trapezoid the opposite angles are supplementary.

65. If the angles at the base of a trapezoid are equal, the other angles are equal, and the trapezoid is isosceles.

66. The diagonals of an isosceles trapezoid are equal.



67. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

HINT. Draw CE and $DF \perp$ to CD. Show that $\triangle s$ ADF and BCE are equal, that $\triangle s$ COD and AOB are isosceles, and that $\triangle s$ AOC and BOD are equal.

68. $ABCD$ is a parallelogram, E and F the middle points of AD and BC respectively: show that BE and DF will trisect the diagonal AC.

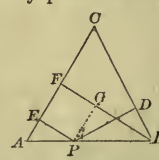
69. If from the diagonal BD of a square $ABCD$, BE is cut off equal to BC, and EF is drawn perpendicular to BD to meet DC at F, show that DE is equal to EF, and also to FC.

70. The bisector of the vertical angle A of a triangle ABC, and the bisectors of the exterior angles at the base formed by producing the sides AB and AC, meet in a point which is equidistant from the base and the sides produced.

71. If the two angles at the base of a triangle are bisected, and through the point of meeting of the bisectors a line is drawn parallel to the base, the length of this parallel between the sides is equal to the sum of the segments of the sides between the parallel and the base.

72. If one of the acute angles of a right triangle is double the other, the hypotenuse is double the shortest side.

73. The sum of the perpendiculars dropped from any point in the base of an isosceles triangle to the legs is constant, and equal to the altitude upon one of the legs.



HINT. Let PD and PE be the two ⊥s, BF the altitude upon AC. Draw $PG \perp$ to BF, and prove the △s PBG and PBD equal.

74. The sum of the perpendiculars dropped from any point within an equilateral triangle to the three sides is constant, and equal to the altitude.

HINT. Draw through the point a line ∥ to the base, and apply Ex. 73.

75. What is the locus of all points equidistant from a pair of intersecting lines?

76. In the triangle CAB the bisector of the angle C makes with the perpendicular from C to AB an angle equal to half the difference of the angles A and B.

77. If one angle of an isosceles triangle is equal to 60°, the triangle is equilateral.

BOOK II.

THE CIRCLE.

DEFINITIONS.

210. A *circle* is a portion of a plane bounded by a curved line called a *circumference*, all points of which are equally distant from a point within called the *centre*.

211. A *radius* is a straight line drawn from the centre to the circumference; and a *diameter* is a straight line drawn through the centre, having its extremities in the circumference.

By the definition of a circle, all its radii are equal. All its diameters are equal, since the diameter is equal to two radii.

212. A *secant* is a straight line which intersects the circumference in two points; as, AD, Fig. 1.

213. A *tangent* is a straight line which touches the circumference but does not intersect it; as, BC, Fig. 1. The point in which the tangent touches the circumference is called the *point of contact*, or *point of tangency*.

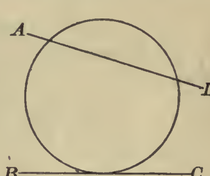


FIG. 1.

214. Two *circumferences* are tangent to each other when they are both tangent to a straight line at the same point; and are tangent *internally* or *externally*, according as one circumference lies wholly *within* or *without* the other.

215. An *arc* of a circle is any portion of the circumference. An arc equal to one-half the circumference is called a *semicircumference.*

216. A *chord* is a straight line having its extremities in the circumference.

Every chord subtends two arcs whose sum is the circumference; thus, the chord AB (Fig. 3) subtends the smaller arc AB and the larger arc $BCDEA$. If a chord and its arc are spoken of, the less arc is meant unless it is otherwise stated.

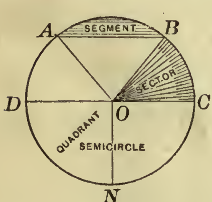


FIG. 2.

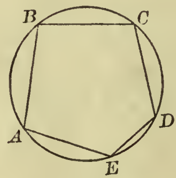


FIG. 3.

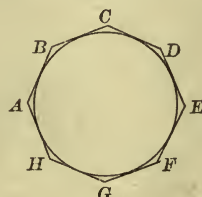


FIG. 4.

217. A *segment* of a circle is a portion of a circle bounded by an arc and its chord.

A segment equal to one-half the circle is called a *semicircle.*

218. A *sector* of a circle is a portion of the circle bounded by two radii and the arc which they intercept.

A sector equal to one-fourth of the circle is called a *quadrant.*

219. A straight line is *inscribed in a circle* if it is a chord.

220. An angle is *inscribed in a circle* if its vertex is in the circumference and its sides are chords.

221. An angle is *inscribed in a segment* if its vertex is on the arc of the segment and its sides pass through the extremities of the arc.

222. A polygon is *inscribed in a circle* if its sides are chords of the circle.

223. A circle is *inscribed in a polygon* if the circumference touches the sides of the polygon but does not intersect them.

224. A polygon is *circumscribed about a circle* if all the sides of the polygon are tangents to the circle.

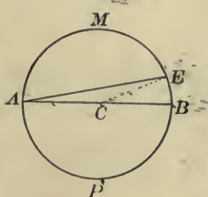
225. A circle is *circumscribed about a polygon* if the circumference passes through all the vertices of the polygon.

226. Two circles are equal if they have equal radii; for they will coincide if one is applied to the other; conversely, two equal circles have equal radii.

Two circles are concentric if they have the same centre.

PROPOSITION I. THEOREM.

227. *The diameter of a circle is greater than any other chord; and bisects the circle and the circumference.*



Let AB be the diameter of the circle $AMBP$, and AE any other chord.

To prove $AB > AE$, and that AB bisects the circle and the circumference.

Proof. I. From C, the centre of the ⊙, draw CE.

$$CE = CB,$$

(being radii of the same circle).

But $AC + CE > AE$, § 137

(the sum of two sides of a △ is > the third side).

Then $AC + CB > AE$, or $AB > AE$. Ax. 9

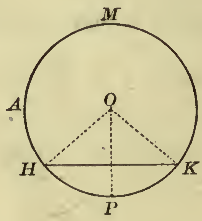
II. Fold over the segment AMB on AB as an axis until it falls upon APB, § 59. The points A and B will remain fixed; therefore the arc AMB will coincide with the arc APB; because all points in each are equally distant from the centre C. § 210

Hence the two figures coincide throughout and are equal. § 59

Q. E. D.

Proposition II. Theorem.

228. *A straight line cannot intersect the circumference of a circle in more than two points.*



***Let** HK **be any line cutting the circumference** AMP.*

To prove that HK can intersect the circumference in only two points.

Proof. If possible, let HK intersect the circumference in three points H, P, and K.

From O, the centre of the ⊙, draw OH, OP, and OK.

Then OH, OP, and OK are equal,

(being radii of the same circle).

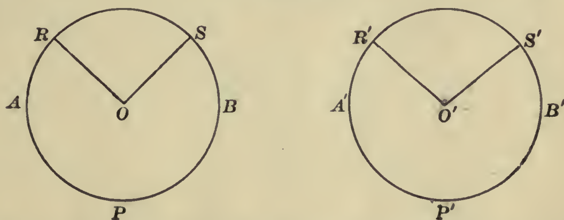
Hence, we have three equal straight lines OH, OP, and OK drawn from the same point to a given straight line. But this is impossible, § 120

(only two equal straight lines can be drawn from a point to a straight line).

Therefore, HK can intersect the circumference in only two points. Q. E. D.

Proposition III. Theorem.

229. *In the same circle, or equal circles, equal angles at the centre intercept equal arcs;* CONVERSELY, *equal arcs subtend equal angles at the centre.*



In the equal circles ABP and $A'B'P'$ let $\angle O = \angle O'$.

To prove arc $RS =$ arc $R'S'$.

Proof. Apply $\odot$ ABP to $\odot$ $A'B'P'$,

so that $\angle O$ shall coincide with $\angle O'$.

R will fall upon R', and S upon S', § 226

(*for $OR = O'R'$, and $OS = O'S'$, being radii of equal* ⓢ).

Then the arc RS will coincide with the arc $R'S'$,

since all points in the arcs are equidistant from the centre. § 210

$\therefore$ arc $RS =$ arc $R'S'$.

CONVERSELY: **Let arc $RS =$ arc $R'S'$.**

To prove $\angle O = \angle O'$.

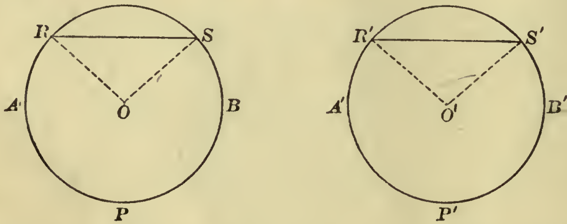
Proof. Apply $\odot$ ABP to $\odot$ $A'B'P'$, so that arc RS shall fall upon arc $R'S'$, R falling upon R', S upon S', and O upon O'.

Then RO will coincide with $R'O'$, and SO with $S'O'$.

$\therefore$ ∠s O and O' coincide and are equal. Q. E. D.

Proposition IV. Theorem.

230. *In the same circle, or equal circles, if two chords are equal, the arcs which they subtend are equal;* CONVERSELY, *if two arcs are equal, the chords which subtend them are equal.*



In the equal circles *ABP* and *A′B′P′*, let chord *RS* = chord *R′S′*.

To prove arc $RS =$ arc $R'S'$.

Proof. Draw the radii OR, OS, $O'R'$, and $O'S'$.

In the △ ORS and $O'R'S'$

$RS = R'S'$, Hyp.

the radii OR and $OS =$ the radii $O'R'$ and $O'S'$. § 226

$\therefore \triangle ROS = \triangle R'O'S'$, § 160

(*three sides of the one being equal to three sides of the other*).

$\therefore \angle O = \angle O'$,

$\therefore$ arc $RS =$ arc $R'S'$, § 229

(*in equal ⊙, equal △ at the centre intercept equal arcs*).

Q. E. D.

CONVERSELY: **Let arc *RS* = arc *R′S′*.**

To prove chord $RS =$ chord $R'S'$.

Proof. $\angle O = \angle O'$, § 229

(*equal arcs in equal ⊙ subtend equal △ at the centre*),

and OR and $OS = O'R'$ and $O'S'$, respectively. § 226

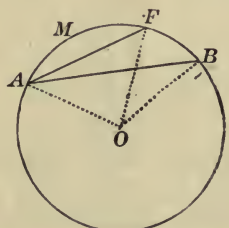
$\therefore \triangle ORS = \triangle O'R'S'$, § 150

(*having two sides equal each to each and the included △ equal*).

$\therefore$ chord $RS =$ chord $R'S'$. Q. E. D.

Proposition V. Theorem.

231. *In the same circle, or equal circles, if two arcs are unequal, and each is less than a semi-circumference, the greater arc is subtended by the greater chord;* CONVERSELY, *the greater chord subtends the greater arc.*



In the circle whose centre is O, let the arc AMB be greater than the arc AMF.

To prove chord AB greater than chord AF.

Proof. Draw the radii OA, OF, and OB.

Since F is between A and B, OF will fall between OA and OB, and $\angle AOB$ be greater than $\angle AOF$.

Hence, in the $\triangle$ AOB and AOF,

the radii OA and $OB =$ the radii OA and OF,

but $\angle AOB$ is greater than $\angle AOF$.

$\therefore AB > AF$, § 152

(*the $\triangle$ having two sides equal each to each, but the included $\angle$ unequal*).

CONVERSELY: **Let AB be greater than AF.**

To prove arc AB greater than arc AF.

In the $\triangle$ AOB and AOF,

OA and $OB = OA$ and OF respectively.

But AB is greater than AF. Hyp.

$\therefore \angle AOB$ is greater than $\angle AOF$, § 153

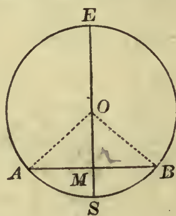
(*the $\triangle$ having two sides equal each to each, but the third sides unequal*).

$\therefore OB$ falls without OF.

$\therefore$ arc AB is greater than arc AF. Q. E. D.

Proposition VI. Theorem.

232. *The radius perpendicular to a chord bisects the chord and the arc subtended by it.*



Let AB be the chord, and let the radius OS be perpendicular to AB at M.

To prove $AM = BM$, *and arc* $AS =$ *arc* BS.

Proof. Draw OA and OB from O, the centre of the circle.

In the rt. $\triangle$ OAM and OBM

the radius $OA =$ the radius OB,

and $OM = OM$. Iden.

$\therefore \triangle OAM = \triangle OBM$, § 161

(*having the hypotenuse and a side of one equal to the hypotenuse and a side of the other*).

$\therefore AM = BM$,

and $\angle AOS = \angle BOS$.

$\therefore$ arc $AS =$ arc BS,

(*equal $\angle$ at the centre intercept equal arcs on the circumference*).

Q. E. D.

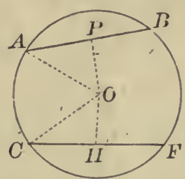
233. Cor. 1. *The perpendicular erected at the middle of a chord passes through the centre of the circle.* For the centre is equidistant from the extremities of a chord, and is therefore in the perpendicular erected at the middle of the chord. § 122

234. Cor. 2. *The perpendicular erected at the middle of a chord bisects the arcs of the chord.*

235. Cor. 3. *The locus of the middle points of a system of parallel chords is the diameter perpendicular to them.*

Proposition VII. Theorem.

236. *In the same circle, or equal circles, equal chords are equally distant from the centre;* AND CONVERSELY.



Let AB and CF be equal chords of the circle $ABFC$.

To prove AB and CF equidistant from the centre O.

Proof. Draw $OP \perp$ to AB, $OH \perp$ to CF, and join OA and OC.

OP and OH bisect AB and CF, § 232

(a radius $\perp$ to a chord bisects it).

Hence, in the rt. $\triangle$ OPA and OHC

$AP = CH$, Ax. 7

the radius OA = the radius OC.

$\therefore \triangle OPA = \triangle OHC$, § 161

(having a side and hypotenuse of the one equal to a side and hypotenuse of the other).

$\therefore OP = OH$.

$\therefore AB$ and CF are equidistant from O.

Conversely: **Let** $OP = OH$.

To prove $AB = CF$.

Proof. In the rt. $\triangle$ OPA and OHC

the radius OA = the radius OC, and $OP = OH$ (by hyp.).

$\therefore \triangle$ OPA and OHC are equal. § 161

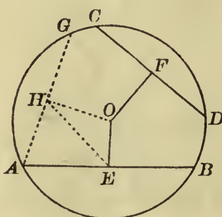
$\therefore AP = CH$.

$\therefore AB = CF$. Ax. 6.

Q. E. D.

Proposition VIII. Theorem.

237. *In the same circle, or equal circles, if two chords are unequal, they are unequally distant from the centre, and the greater is at the less distance.*



In the circle whose centre is *O*, let the chords *AB* and *CD* be unequal, and *AB* the greater; and let *OE* and *OF* be perpendicular to *AB* and *CD* respectively.

To prove $OE < OF.$

Proof. Suppose AG drawn equal to CD, and $OH \perp$ to AG.

Then $OH = OF$, § 236

(in the same ⊙ two equal chords are equidistant from the centre).

Join EH.

OE and OH bisect AB and AG, respectively, § 232

(a radius ⊥ to a chord bisects it).

Since, by hypothesis, AB is greater than CD or its equal AG, AE, the half of AB, is greater than AH, the half of AG.

$\therefore$ the $\angle\, AHE$ is greater than the $\angle\, AEH$, § 158

(the greater of two sides of a △ has the greater ∠ opposite to it).

Therefore, the $\angle\, OHE$, the complement of the $\angle\, AHE$, is less than the $\angle\, OEH$, the complement of the $\angle\, AEH$.

$\therefore OE < OH$, § 159

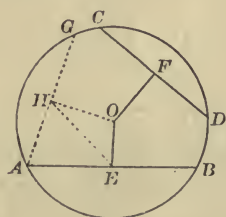
(the greater of two ∠s of a △ has the greater side opposite to it).

$\therefore OE < OF$, the equal of OH.

Q. E. D.

PROPOSITION IX. THEOREM.

238. CONVERSELY: *In the same circle, or equal circles, if two chords are unequally distant from the centre, they are unequal, and the chord at the less distance is the greater.*



In the circle whose centre is O, let AB and CD be unequally distant from O; and let OE perpendicular to AB be less than OF perpendicular to CD.

To prove $AB > CD$.

Proof. Suppose AG drawn equal to CD, and $OH \perp$ to AG.

Then $OH = OF$, § 236

(*in the same* ⊙ *two equal chords are equidistant from the centre*).

Hence, $OE < OH$.

Join EH.

In the $\triangle$ OEH the $\angle$ OHE is less than the $\angle$ OEH, § 158

(*the greater of two sides of a* $\triangle$ *has the greater* $\angle$ *opposite to it*).

Therefore, the $\angle$ AHE, the complement of the $\angle$ OHE, is greater than the $\angle$ AEH, the complement of the $\angle$ OEH.

$\therefore AE > AH$, § 159

(*the greater of two* $\angle$s *of a* $\triangle$ *has the greater side opposite to it*).

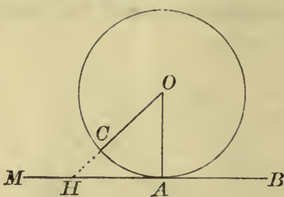
But $AE = \frac{1}{2} AB$, and $AH = \frac{1}{2} AG$.

$\therefore AB > AG$; hence $AB > CD$, the equal of AG.

Q. E. D.

Proposition X. Theorem.

239. *A straight line perpendicular to a radius at its extremity is a tangent to the circle.*



Let MB **be perpendicular to the radius** OA **at** A.

To prove MB *tangent to the circle.*

Proof. From O draw any other line to MB, as OCH.

$$OH > OA, \qquad \S\ 114$$

(*a* $\perp$ *is the shortest line from a point to a straight line*).

$\therefore$ the point H is without the circle.

Hence, every point, except A, of the line MB is without the circle, and therefore MB is a tangent to the circle at A. § 213

Q. E. D.

240. Cor. 1. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.* For, if MB is tangent to the circle at A, every point of MB, except A, is without the circle. Hence, OA is the shortest line from O to MB, and is therefore perpendicular to MB (§ 114); that is, MB is perpendicular to OA.

241. Cor. 2. *A perpendicular to a tangent at the point of contact passes through the centre of the circle.* For a radius is perpendicular to a tangent at the point of contact, and therefore, by § 89, a perpendicular erected at the point of contact coincides with this radius and passes through the centre.

242. Cor. 3. *A perpendicular let fall from the centre of a circle upon a tangent to the circle passes through the point of contact.*

Proposition XI. Theorem.

243. *Parallels intercept equal arcs on a circumference.*

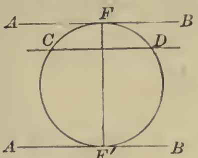


Fig. 1.

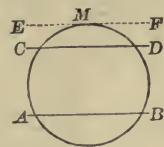


Fig. 2.

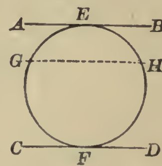


Fig. 3.

Let AB and CD be the two parallels.

Case I. *When AB is a tangent, and CD a secant.* Fig. 1.

Suppose AB touches the circle at F.

To prove arc $CF =$ arc DF.

Proof. Suppose FF' drawn $\perp$ to AB.

This $\perp$ to AB at F is a diameter of the circle. § 241

It is also $\perp$ to CD. § 102

$\therefore$ arc $CF =$ arc DF, § 232

(*a radius $\perp$ to a chord bisects the chord and its subtended arc*).

Also, arc $FCF' =$ arc FDF', § 227

$\therefore$ arc $(FCF' - FC) =$ arc $(FDF' - FD)$, § 82

that is, arc $CF' =$ arc DF'.

Case II. *When AB and CD are secants.* Fig. 2.

Suppose EF drawn $\parallel$ to CD and tangent to the circle at M.

Then arc $AM =$ arc BM

and arc $CM =$ arc DM Case I.

$\therefore$ by subtraction, arc $AC =$ arc BD

Case III. *When AB and CD are tangents.* Fig. 3.

Suppose AB tangent at E, CD at F, and $GH \parallel$ to AB.

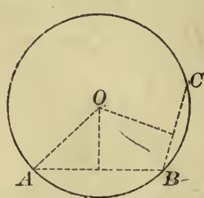
Then arc $GE =$ arc EH Case I.

and arc $GF =$ arc HF

$\therefore$ by addition, arc $EGF =$ arc EHF Q. E. D.

Proposition XII. Theorem.

244. *Through three points not in a straight line, one circumference, and only one, can be drawn.*



Let A, B, C be three points not in a straight line.

To prove that a circumference can be drawn through A, B, and C, and only one.

Proof. Join AB and BC.

At the middle points of AB and BC suppose ⊥s erected.

Since BC is not the prolongation of AB, these ⊥s will intersect in some point O.

The point O, being in the ⊥ to AB at its middle point, is equidistant from A and B; and being in the ⊥ to BC at its middle point, is equidistant from B and C, § 122

(*every point in the perpendicular bisector of a straight line is equidistant from the extremities of the straight line*).

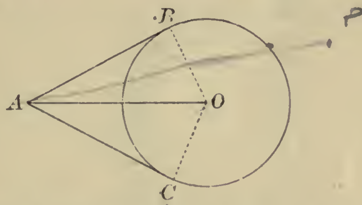
Therefore O is equidistant from A, B, and C; and a circumference described from O as a centre, and with a radius OA, will pass through the three given points.

Only one circumference can be made to pass through these points. For the centre of a circumference passing through the three points must be in both perpendiculars, and hence at their intersection. As two straight lines can intersect in only one point, O is the centre of the only circumference that can pass through the three given points. Q. E. D.

245. Cor. *Two circumferences can intersect in only two points.* For, if two circumferences have three points common, they coincide and form one circumference.

PROPOSITION XIII. THEOREM.

246. *The tangents to a circle drawn from an exterior point are equal, and make equal angles with the line joining the point to the centre.*



Let AB and AC be tangents from A to the circle whose centre is O, and AO the line joining A to O.

To prove $AB = AC$, and $\angle BAO = \angle CAO$.

Proof. Draw OB and OC.

AB is $\perp$ to OB, and $AC \perp$ to OC, § 240

(*a tangent to a circle is $\perp$ to the radius drawn to the point of contact*).

In the rt. $\triangle$s OAB and OAC

$$OB = OC,$$

(*radii of the same circle*).

$$OA = OA. \quad \text{Iden.}$$

$$\therefore \triangle OAB = \triangle OAC, \quad \S\, 161$$

(*having a side and hypotenuse of the one equal to a side and hypotenuse of the other*).

$$\therefore AB = AC,$$

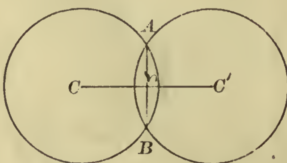
and $$\angle BAO = \angle CAO. \quad \text{Q. E. D.}$$

247. DEF. The line joining the centres of two circles is called the *line of centres*.

248. DEF. A common tangent to two circles is called a *common exterior tangent* when it does not cut the line of centres, and a *common interior tangent* when it cuts the line of centres.

Proposition XIV. Theorem.

249. *If two circumferences intersect each other, the line of centres is perpendicular to their common chord at its middle point.*



Let C and C' be the centres of two circumferences which intersect at A and B. Let AB be their common chord, and CC' join their centres.

To prove $CC' \perp$ *to* AB *at its middle point.*

Proof. A $\perp$ drawn through the middle of the chord AB passes through the centres C and C', § 233

(*a $\perp$ erected at the middle of a chord passes through the centre of the* $\odot$).

$\therefore$ the line CC', having two points in common with this $\perp$, must coincide with it.

$\therefore$ CC' is $\perp$ to AB at its middle point. Q. E. D.

Ex. 78. Describe the relative position of two circles if the line of centres:

(i.) is greater than the sum of the radii;

(ii.) is equal to the sum of the radii;

(iii.) is less than the sum but greater than the difference of the radii;

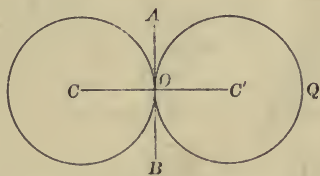
(iv.) is equal to the difference of the radii;

(v.) is less than the difference of the radii.

Illustrate each case by a figure.

Proposition XV. Theorem.

250. *If two circumferences are tangent to each other, the line of centres passes through the point of contact.*



Let the two circumferences, whose centres are C and C', touch each other at O, in the straight line AB, and let CC' be the straight line joining their centres.

To prove O *is in the straight line* CC'.

Proof. A $\perp$ to AB, drawn through the point O, passes through the centres C and C', § 241

(*a $\perp$ to a tangent at the point of contact passes through the centre of the circle*).

$\therefore$ the line CC', having two points in common with this $\perp$ must coincide with it.

$\therefore$ O is in the straight line CC'. Q. E. D.

Ex. 79. The line joining the centre of a circle to the middle of a chord is perpendicular to the chord.

Ex. 80. The tangents drawn through the extremities of a diameter are parallel.

Ex. 81. The perimeter of an inscribed equilateral triangle is equal to half the perimeter of the circumscribed equilateral triangle.

Ex. 82. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

Measurement.

251. To *measure* a quantity of any kind is to find how *many times* it contains another known quantity of the *same kind.*

Thus, to measure a line is to find how many times it contains another known line, called the *linear unit.*

The number which expresses how many times a quantity contains the unit, joined with the name of the unit, is called the *numerical measure* of that quantity; as, 5 yards, etc.

252. The magnitude of a quantity is always *relative* to the magnitude of another quantity of the *same kind.* No quantity is great or small except by comparison. This relative magnitude is called their *ratio*, and is expressed by the indicated quotient of their numerical measures when the same unit of measure is applied to both.

The ratio of a to b is written $\frac{a}{b}$, or $a : b$.

253. Two quantities that can be expressed in integers in terms of a common unit are said to be *commensurable.* The common unit is called a *common measure*, and each quantity is called a *multiple* of this common measure.

Thus, a common measure of $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet is $\frac{1}{6}$ of a foot, which is contained 15 times in $2\frac{1}{2}$ feet, and 22 times in $3\frac{2}{3}$ feet. Hence, $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet are multiples of $\frac{1}{6}$ of a foot, $2\frac{1}{2}$ feet being obtained by taking $\frac{1}{6}$ of a foot 15 times, and $3\frac{2}{3}$ by taking $\frac{1}{6}$ of a foot 22 times.

254. When two quantities are *incommensurable*, that is, have no common unit in terms of which *both* quantities can be expressed in *integers*, it is impossible to find a fraction that will indicate the exact value of the ratio of the given quantities. It is possible, however, by taking the unit sufficiently small, to find a fraction that shall differ from the true value of the ratio by as little as we please.

Thus, suppose a and b to denote two lines, such that

a ————————

b ———————— $\frac{a}{b} = \sqrt{2}.$

Now $\sqrt{2} = 1.41421356\ldots\ldots$, a value greater than 1.414213, but less than 1.414214.

If, then, a *millionth part* of b be taken as the unit, the value of the ratio $\frac{a}{b}$ lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these fractions by less than $\frac{1}{1000000}$.

By carrying the decimal farther, a fraction may be found that will differ from the true value of the ratio by less than a *billionth, a trillionth, or any other assigned value whatever.*

Expressed generally, when a and b are incommensurable, and b is divided into any integral number (n) of equal parts, if one of these parts is contained in a more than m times, but less than $m + 1$ times, then

$$\frac{a}{b} > \frac{m}{n}, \text{ but } < \frac{m+1}{n};$$

that is, the value of $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$.

The error, therefore, in taking either of these values for $\frac{a}{b}$ is less than $\frac{1}{n}$. But by *increasing n indefinitely*, $\frac{1}{n}$ can be made *to decrease indefinitely*, and to become *less than any assigned value, however small*, though it cannot be made *absolutely equal to zero.*

Hence, the ratio of two incommensurable quantities cannot be expressed *exactly* by figures, but it may be expressed *approximately* within any assigned measure of precision.

255. The ratio of two incommensurable quantities is called an *incommensurable ratio;* and is *a fixed value* toward which its successive approximate values constantly tend.

256. THEOREM. *Two incommensurable ratios are equal if, when the unit of measure is indefinitely diminished, their approximate values constantly remain equal.*

Let $a:b$ and $a':b'$ be two incommensurable ratios whose true values lie between the approximate values $\frac{m}{n}$ and $\frac{m+1}{n}$, when the unit of measure is indefinitely diminished. Then they cannot differ so much as $\frac{1}{n}$.

Now the difference (if any) between the fixed values $a:b$ and $a':b'$, is a *fixed value*. Let d denote this difference.

Then $$d < \frac{1}{n}.$$

But if d has *any* value, *however small*, $\frac{1}{n}$, which by hypothesis can be indefinitely diminished, can be made less than d.

Therefore d cannot have any value; that is, $d = 0$, and there is no difference between the ratios $a:b$ and $a':b'$; therefore $a:b = a':b'$.

THE THEORY OF LIMITS.

257. When a quantity is regarded as having a *fixed* value throughout the same discussion, it is called a *constant;* but when it is regarded, under the conditions imposed upon it, as having *different successive* values, it is called a *variable.*

When it can be shown that the value of a variable, measured at a series of definite intervals, can by continuing the series be made to differ from a given constant by less than any assigned quantity, however small, but cannot be made absolutely equal to the constant, that constant is called the *limit* of the variable, and the variable is said to *approach indefinitely to its limit.*

If the variable is increasing, its limit is called a *superior* limit; if decreasing, an *inferior* limit.

Suppose a point to move from A toward B, under the conditions that the first second it shall move

one-half the distance from A to B, that is, to M; the next second, one-half the remaining distance, that is, to M'; the next second, one-half the remaining distance, that is, to M''; and so on indefinitely.

Then it is evident that the moving point *may approach as near to B as we please, but will never arrive at B.* For, however near it may be to B at any instant, the next second it will pass over one-half the interval still remaining; it must, therefore, approach nearer to B, since *half* the interval still remaining is *some* distance, but will not reach B, since *half* the interval still remaining is not the *whole* distance.

Hence, the distance from A to the moving point is an increasing variable, which indefinitely approaches the constant AB as its *limit;* and the distance from the moving point to B is a decreasing variable, which indefinitely approaches the constant zero as its *limit.*

If the length of AB be two inches, and the variable be denoted by x, and the difference between the variable and its limit, by v:

after one second,	$x = 1$,	$v = 1$;
after two seconds,	$x = 1 + \frac{1}{2}$,	$v = \frac{1}{2}$;
after three seconds,	$x = 1 + \frac{1}{2} + \frac{1}{4}$,	$v = \frac{1}{4}$;
after four seconds,	$x = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$,	$v = \frac{1}{8}$;

and so on indefinitely.

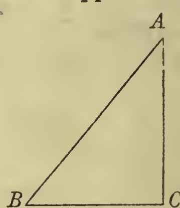
Now the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, etc., is less than 2; but by taking a great number of terms, the sum can be made to differ from 2 by as little as we please. Hence 2 is the limit of the sum of the series, when the number of the terms is increased indefinitely; and 0 is the limit of the difference between this variable sum and 2.

Consider the repetend 0.33333....., which may be written

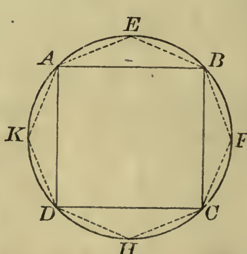
$$\tfrac{3}{10}+\tfrac{3}{100}+\tfrac{3}{1000}+\tfrac{3}{10000}+\cdots$$

However great the number of terms of this series we take, the sum of these terms will be less than $\frac{1}{3}$; but the more terms we take the nearer does the sum approach $\frac{1}{3}$. Hence the sum of the series, as the number of terms is increased, approaches indefinitely the constant $\frac{1}{3}$ as a limit.

258. In the right triangle ACB, if the vertex A approaches indefinitely the base BC, the angle B diminishes, and approaches zero indefinitely; if the vertex A moves away from the base indefinitely, the angle B increases and approaches a right angle indefinitely; but B cannot become zero or a right angle, so long as ACB is a triangle; for if B becomes zero, the triangle becomes the straight line BC, and if B becomes a right angle, the triangle becomes two parallel lines AC and AB perpendicular to BC. Hence the value of B must lie between 0° and 90° as limits.



259. Again, suppose a square $ABCD$ inscribed in a circle, and E, F, H, K the middle points of the arcs subtended by the sides of the square. If we draw the straight lines AE, EB, BF, etc., we shall have an inscribed polygon of double the number of sides of the square.



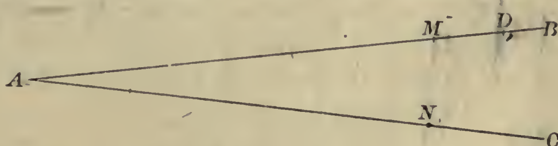
The length of the perimeter of this polygon, represented by the dotted lines, is greater than that of the square, since two sides replace each side of the square and form with it a triangle, and two sides of a triangle are together greater than the third side; but less than the length of the circumference, for it is made up of

straight lines, each one of which is less than the part of the circumference between its extremities.

By continually repeating the process of doubling the number of sides of each resulting inscribed figure, the length of the perimeter will increase with the increase of the number of sides; but it cannot become equal to the length of the circumference, for the perimeter will continue to be made up of straight lines, each one of which is less than the part of the circumference between its extremities.

The length of the circumference is therefore the *limit* of the length of the perimeter as the number of sides of the inscribed figure is indefinitely increased.

260. THEOREM. *If two variables are constantly equal and each approaches a limit, their limits are equal.*



Let AM and AN be two variables which are constantly equal and which approach indefinitely AB and AC respectively as limits.

To prove $AB = AC$.

Proof. If possible, suppose $AB > AC$, and take $AD = AC$.

Then the variable AM may assume values between AD and AB, while the variable AN must always be less than AD. But this is contrary to the hypothesis that the variables should continue equal.

$$\therefore AB \text{ cannot be } > AC.$$

In the same way it may be proved that AC cannot be $> AB$.

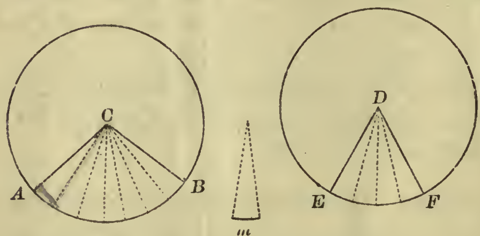
$\therefore AB$ and AC are two values neither of which is greater than the other.

Hence $AB = AC$. Q. E. D.

Measure of Angles.

Proposition XVI. Theorem.

261. *In the same circle, or equal circles, two angles at the centre have the same ratio as their intercepted arcs.*



Case I. *When the arcs are commensurable.*

In the circles whose centres are C and D, let ACB and EDF be the angles, AB and EF the intercepted arcs.

To prove $$\frac{\angle ACB}{\angle EDF} = \frac{\text{arc } AB}{\text{arc } EF}.$$

Proof. Let m be a common measure of AB and EF.

Suppose m to be contained in AB seven times, and in EF four times.

Then $$\frac{\text{arc } AB}{\text{arc } EF} = \frac{7}{4}. \qquad (1)$$

At the several points of division on AB and EF draw radii. These radii will divide $\angle ACB$ into seven parts, and $\angle EDF$ into four parts, equal each to each, § 229

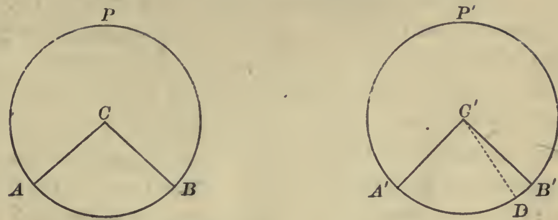
(*in the same ⊙, or equal ⊚, equal arcs subtend equal ∠s at the centre*).

$$\therefore \frac{\angle ACB}{\angle EDF} = \frac{7}{4}. \qquad (2)$$

From (1) and (2),

$$\frac{\angle ACB}{\angle EDF} = \frac{\text{arc } AB}{\text{arc } EF}. \qquad \text{Ax. 1}$$

CASE II. *When the arcs are incommensurable.*



In the equal circles ABP and A'B'P' let the angles ACB and A'C'B' intercept the incommensurable arcs AB and A'B'.

To prove $$\frac{\angle ACB}{\angle A'C'B'} = \frac{\text{arc } AB}{\text{arc } A'B'}.$$

Proof. Divide AB into any number of equal parts, and apply one of these parts as a unit of measure to $A'B'$ as many times as it will be contained in $A'B'$.

Since AB and $A'B'$ are incommensurable, a certain number of these parts will extend from A' to some point, as D, leaving a remainder DB' less than one of these parts.

Draw $C'D$.

Since AB and $A'D$ are commensurable,

$$\frac{\angle ACB}{\angle A'C'D} = \frac{\text{arc } AB}{\text{arc } A'D}. \qquad \text{Case I.}$$

If the unit of measure is indefinitely diminished, these ratios continue equal, and approach indefinitely the limiting ratios

$$\frac{\angle ACB}{\angle A'C'B'} \text{ and } \frac{\text{arc } AB}{\text{arc } A'B'}.$$

Therefore $$\frac{\angle ACB}{\angle A'C'B'} = \frac{\text{arc } AB}{\text{arc } A'B'}. \qquad \S\ 260$$

(*If two variables are constantly equal, and each approaches a limit, their limits are equal.*)

Q. E. D.

262. The circumference, like the angular magnitude about a point, is divided into 360 equal parts, called *degrees.* The arc-degree is subdivided into 60 equal parts, called *minutes;* and the minute into 60 equal parts, called *seconds.*

Since an angle at the centre has the same number of angle-degrees, minutes, and seconds as the intercepted arc has of arc-degrees, minutes, and seconds, we say: *An angle at the centre is measured by its intercepted arc;* meaning, *An angle at the centre is such a part of the whole angular magnitude about the centre as its intercepted arc is of the whole circumference.*

Proposition XVII. Theorem.

263. ***An inscribed angle is measured by one-half of the arc intercepted between its sides.***

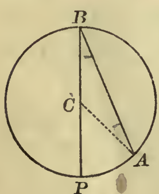


Fig. 1.

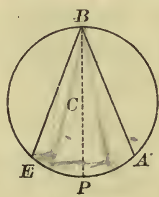


Fig. 2.

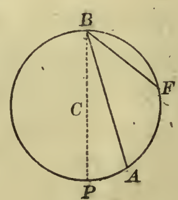


Fig. 3.

Case I. *When one side of the angle is a diameter.*

In the circle PAB (Fig. 1), let the centre C be in one of the sides of the inscribed angle B.

To prove $\angle B$ *is measured by* $\frac{1}{2}$ *arc* PA.

Proof. Draw CA.

Radius CA = radius CB.

$\therefore \angle B = \angle A$, § 154

(*being opposite equal sides of the* $\triangle$ CAB).

But $\angle PCA = \angle B + \angle A$, § 145

(*the exterior* $\angle$ *of a* $\triangle$ *is equal to the sum of the two opposite interior* $\angle$s).

$\therefore \angle PCA = 2 \angle B$.

But $\angle PCA$ is measured by PA, § 262

(*the* $\angle$ *at the centre is measured by the intercepted arc*).

$\therefore \angle B$ is measured by $\frac{1}{2}$ PA.

CASE II. *When the centre is within the angle.*

In the circle BAE (Fig. 2), let the centre C fall within the angle EBA.

To prove $\angle EBA$ *is measured by* $\frac{1}{2}$ *arc* EA.

Proof. Draw the diameter BCP.

$\angle PBA$ is measured by $\frac{1}{2}$ arc PA, Case I.

$\angle PBE$ is measured by $\frac{1}{2}$ arc PE. Case I.

$\therefore \angle PBA + \angle PBE$ is measured by $\frac{1}{2}$ (arc PA + arc PE),

or $\angle EBA$ is measured by $\frac{1}{2}$ arc EA.

CASE III. *When the centre is without the angle.*

In the circle BFP (Fig. 3), let the centre C fall without the angle ABF.

To prove $\angle ABF$ *is measured by* $\frac{1}{2}$ *arc* AF.

Proof. Draw the diameter BCP.

$\angle PBF$ is measured by $\frac{1}{2}$ arc PF, Case I.

$\angle PBA$ is measured by $\frac{1}{2}$ arc PA. Case I.

$\therefore \angle PBF - \angle PBA$ is measured by $\frac{1}{2}$ (arc PF − arc PA),

or $\angle ABF$ is measured by $\frac{1}{2}$ arc AF. Q. E. D.

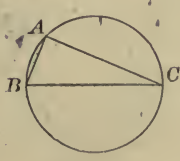


FIG. 1.

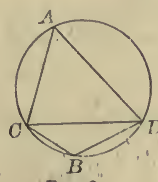


FIG. 2.

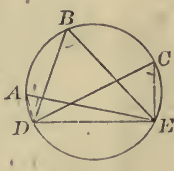


FIG. 3.

264. COR. 1. *An angle inscribed in a semicircle is a right angle.* For it is measured by one-half a semi-circumference.

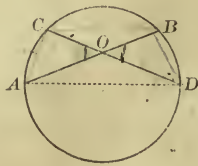
265. COR. 2. *An angle inscribed in a segment greater than a semicircle is an acute angle.* For it is measured by an arc less than half a semi-circumference; as, $\angle CAD$. Fig. 2.

266. COR. 3. *An angle inscribed in a segment less than a semicircle is an obtuse angle.* For it is measured by an arc greater than half a semi-circumference; as, $\angle CBD$. Fig. 2.

267. COR. 4. *All angles inscribed in the same segment are equal.* For they are measured by half the same arc. Fig. 3.

PROPOSITION XVIII. THEOREM.

268. *An angle formed by two chords intersecting within the circumference, is measured by one-half the sum of the intercepted arcs.*



Let the angle AOC be formed by the chords AB and CD.

To prove $\angle AOC$ *is measured by* $\frac{1}{2}(AC + BD)$.

Proof. Draw AD.

$$\angle COA = \angle D + \angle A, \qquad \S\ 145$$

(*the exterior* $\angle$ *of a* $\triangle$ *is equal to the sum of the two opposite interior* $\angle$s).

But $\angle D$ is measured by $\frac{1}{2}$ arc AC, § 263

and $\angle A$ is measured by $\frac{1}{2}$ arc BD,

(*an inscribed* $\angle$ *is measured by* $\frac{1}{2}$ *the intercepted arc*).

$\therefore \angle COA$ is measured by $\frac{1}{2}(AC + BD)$.

Q. E. D.

Ex. 83. The opposite angles of an inscribed quadrilateral are supplements of each other.

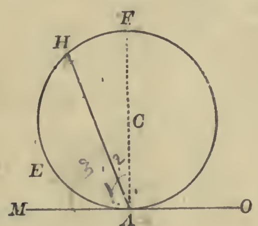
Ex. 84. If through a point within a circle two perpendicular chords are drawn, the sum of the opposite arcs which they intercept is equal to a semi-circumference.

Ex. 85. The line joining the centre of the square described upon the hypotenuse of a rt. $\triangle$, to the vertex of the rt. $\angle$, bisects the right angle.

HINT. Describe a circle upon the hypotenuse as diameter.

Proposition XIX. Theorem.

269. ***An angle formed by a tangent and a chord is measured by one-half the intercepted arc.***



***Let** MAH **be the angle formed by the tangent** MO **and chord** AH.*

To prove $\angle\ MAH$ *is measured by* $\frac{1}{2}$ *arc* AEH.

Proof. Draw the diameter ACF.

$\angle\ MAF$ is a rt. $\angle$, § 240

(*the radius drawn to a tangent at the point of contact is* $\perp$ *to it*).

$\angle\ MAF$ being a rt. $\angle$, is measured by $\frac{1}{2}$ the semi-circumference AEF.

But $\angle\ HAF$ is measured by $\frac{1}{2}$ arc HF, § 263

(*an inscribed* $\angle$ *is measured by* $\frac{1}{2}$ *the intercepted arc*).

$\therefore \angle\ MAF - \angle\ HAF$ is measured by $\frac{1}{2}(AEF - HF)$;

or $\angle\ MAH$ is measured by $\frac{1}{2}\ AEH$.

Q. E. D.

Ex. 86. If two circles touch each other and two secants are drawn through the point of contact, the chords joining their extremities are parallel. Hint. Draw the common tangent.

Proposition XX. Theorem.

270. ***An angle formed by two secants, two tangents, or a tangent and a secant, intersecting without the circumference, is measured by one-half the difference of the intercepted arcs.***

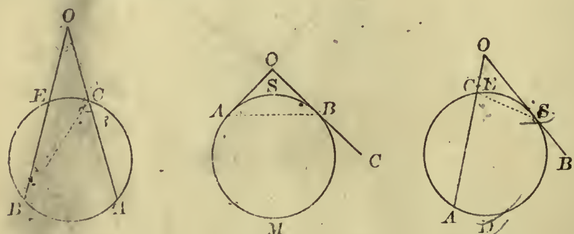


Fig. 1. Fig. 2. Fig. 3.

Case I. *Angle formed by two secants.*

Let the angle O (Fig. 1) be formed by the two secants OA and OB.

To prove $\angle O$ *is measured by* $\frac{1}{2}(AB - EC)$.

Proof. Draw CB.

$$\angle ACB = \angle O + \angle B, \qquad \S\ 145$$

(*the exterior* $\angle$ *of a* $\triangle$ *is equal to the sum of the two opposite interior* $\angle s$).

By taking away $\angle B$ from both sides,

$$\angle O = \angle ACB - \angle B.$$

But $\angle ACB$ is measured by $\frac{1}{2} AB$, § 263

and $\angle B$ is measured by $\frac{1}{2} CE$,

(*an inscribed* $\angle$ *is measured by* $\frac{1}{2}$ *the intercepted arc*).

$\therefore \angle O$ is measured by $\frac{1}{2}(AB - CE)$.

CASE II. *Angle formed by two tangents.*

Let the angle O (Fig. 2) be formed by the two tangents OA and OB.

To prove $\angle O$ *is measured by* $\frac{1}{2}(AMB - ASB)$.

Proof. Draw AB.

$\angle ABC = \angle O + \angle OAB$, § 145

(*the exterior* $\angle$ *of a* $\triangle$ *is equal to the sum of the two opposite interior* $\angle$s).

By taking away $\angle OAB$ from both sides,

$$\angle O = \angle ABC - \angle OAB.$$

But $\angle ABC$ is measured by $\frac{1}{2} AMB$, § 269

and $\angle OAB$ is measured by $\frac{1}{2} ASB$,

(*an* $\angle$ *formed by a tangent and a chord is measured by* $\frac{1}{2}$ *the intercepted arc*).

$\therefore \angle O$ is measured by $\frac{1}{2}(AMB - ASB)$.

CASE III. *Angle formed by a tangent and a secant.*

Let the angle O (Fig. 3) be formed by the tangent OB and the secant OA.

To prove $\angle O$ *is measured by* $\frac{1}{2}(ADS - CES)$.

Proof. Draw CS.

$\angle ACS = \angle O + \angle CSO$, § 145

(*the exterior* $\angle$ *of a* $\triangle$ *is equal to the sum of the two opposite interior* $\angle$s).

By taking away $\angle CSO$ from both sides,

$$\angle O = \angle ACS - \angle CSO.$$

But $\angle ACS$ is measured by $\frac{1}{2} ADS$, § 263

(*being an inscribed* $\angle$),

and $\angle CSO$ is measured by $\frac{1}{2} CES$, § 269

(*being an* $\angle$ *formed by a tangent and a chord*).

$\therefore \angle O$ is measured by $\frac{1}{2}(ADS - CES)$.

Q. E. D.

Problems of Construction.

Proposition XXI. Problem.

271. ***At a given point in a straight line, to erect a perpendicular to that line.***

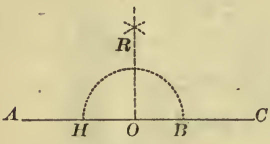


Fig. 1.

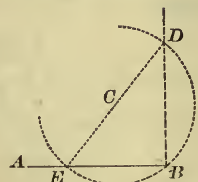


Fig. 2.

I. ***Let O be the given point in AC. (Fig. 1).***

To erect a $\perp$ *to the line AC at the point O.*

Construction. From O as a centre, with any radius OB, describe an arc intersecting AC in two points H and B.

From H and B as centres, with equal radii greater than OB, describe two arcs intersecting at R. Join OR.

Then the line OR is the $\perp$ required.

Proof. Since O and R are two points at equal distances from H and B, they determine the position of a perpendicular to the line HB at its middle point O. § 123

Q. E. F.

II. *When the given point is at the end of the line.*

Let B be the given point. (Fig. 2).

To erect a $\perp$ *to the line AB at B.*

Construction. Take any point C without AB; and from C as a centre, with the distance CB as a radius, describe an arc intersecting AB at E.

Draw EC, and prolong it to meet the arc again at D.

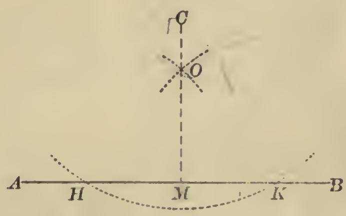
Join BD, and BD is the $\perp$ required.

Proof. The $\angle B$ is inscribed in a semicircle, and is therefore a right angle. § 264

Hence BD is $\perp$ to AB. Q. E. F.

Proposition XXII. Problem.

272. *From a point without a straight line, to let fall a perpendicular upon that line.*



Let AB be a given straight line, and C a given point without the line.

To let fall a ⊥ to the line AB from the point C.

Construction. From C as a centre, with a radius sufficiently great, describe an arc cutting AB in two points, H and K.

From H and K as centres, with equal radii greater than $\frac{1}{2}HK$, describe two arcs intersecting at O.

Draw CO,

and produce it to meet AB at M.

CM is the ⊥ required.

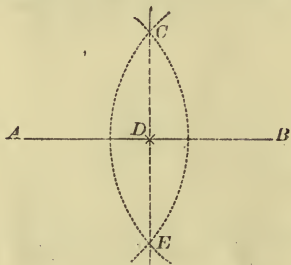
Proof. Since C and O are two points equidistant from H and K, they determine a ⊥ to HK at its middle point. § 123

Q. E. F.

NOTE. *Given* lines of the figures are full lines, *resulting* lines are long-dotted, and *auxiliary* lines are short-dotted.

Proposition XXIII. Problem.

273. *To bisect a given straight line.*



Let AB be the given straight line.

To bisect the line AB.

Construction. From A and B as centres, with equal radii greater than $\frac{1}{2}$ AB, describe arcs intersecting at C and E.

Join CE.

Then the line CE bisects AB.

Proof. C and E are two points equidistant from A and B. Hence they determine a $\perp$ to the middle point of AB. § 123

Q. E. F.

Ex. 87. To find in a given line a point X which shall be equidistant from two given points.

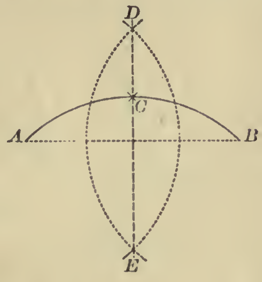
Ex. 88. To find a point X which shall be equidistant from two given points and at a given distance from a third given point.

Ex. 89. To find a point X which shall be at given distances from two given points.

Ex. 90. To find a point X which shall be equidistant from three given points.

Proposition XXIV. Problem.

274. *To bisect a given arc.*



Let ACB be the given arc.

To bisect the arc ACB.

Construction. Draw the chord AB.

From A and B as centres, with equal radii greater than $\frac{1}{2}$ AB, describe arcs intersecting at D and E,

Draw DE.

DE bisects the arc ACB.

Proof. Since D and E are two points equidistant from A and B, they determine a $\perp$ erected at the middle of chord AB. § 123

And a $\perp$ erected at the middle of a chord passes through the centre of the ⊙, and bisects the arc of the chord. § 234

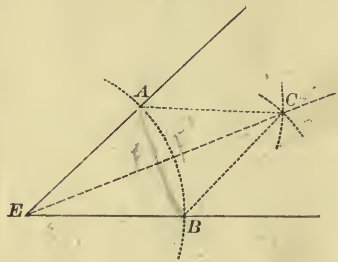
Q. E. F.

Ex. 91. To construct a circle having a given radius and passing through two given points.

Ex. 92. To construct a circle having its centre in a given line and passing through two given points.

Proposition XXV. Problem.

275. *To bisect a given angle.*



Let AEB be the given angle.

To bisect $\angle AEB$.

Construction. From E as a centre, with any radius, as EA, describe an arc cutting the sides of the $\angle E$ at A and B.

From A and B as centres, with equal radii greater than one-half the distance from A to B, describe two arcs intersecting at C.

Join EC, AC, and BC.

EC bisects the $\angle E$.

Proof. In the $\triangle$s AEC and BEC

	$AE = BE$, and $AC = BC$,	Cons.
and	$EC = EC$.	Iden.
	$\therefore \triangle AEC = \triangle BEC$,	§ 160

(having three sides equal each to each).

$\therefore \angle AEC = \angle BEC$.

Q. E. F.

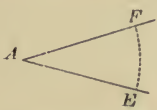
Ex. 93. To divide a right angle into three equal parts.

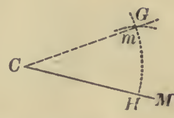
Ex. 94. To construct an equilateral triangle, having given one side.

Ex. 95. To find a point X which shall be equidistant from two given points and also equidistant from two given intersecting lines.

PROPOSITION XXVI. PROBLEM.

276. *At a given point in a given straight line, to construct an angle equal to a given angle.*





Let C be the given point in the given line CM, and A the given angle.

To construct an $\angle$ at C equal to the $\angle$ A.

Construction. From A as a centre, with any radius, as AE, describe an arc cutting the sides of the $\angle$ A at E and F.

From C as a centre, with a radius equal to AE,

describe an arc cutting CM at H.

From H as a centre, with a radius equal to the distance EF,

describe an arc intersecting the arc HG at m.

Draw Cm, and HCm is the required angle.

Proof. The chords EF and Hm are equal. Cons.

$\therefore$ arc $EF =$ arc Hm, § 230

(*in equal Ⓢ equal chords subtend equal arcs*).

$\therefore \angle C = \angle A$, § 229

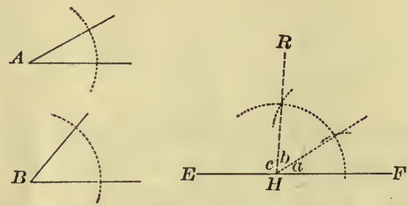
(*in equal Ⓢ equal arcs subtend equal $\angle$s at the centre*). Q. E. F.

Ex. 96. In a triangle ABC, draw DE parallel to the base BC, cutting the sides of the triangle in D and E, so that DE shall equal $DB \pm EC$. P 7 Ex 7

Ex. 97. If an interior point O of a triangle ABC is joined to the vertices B and C, the angle BOC is greater than the angle BAC of the triangle.

Proposition XXVII. Problem.

277. ***Two angles of a triangle being given, to find the third angle.***



Let A and B be the two given angles of a triangle:

To find the third ∠ of the △.

Construction. Take any straight line, as EF, and at any point, as H,

construct ∠ a equal to ∠ A, § 276

and ∠ b equal to ∠ B.

Then ∠ c is the ∠ required.

Proof. Since the sum of the three ∠s of a △ = 2 rt. ∠s, § 138

and the sum of the three ∠s a, b, and c, = 2 rt. ∠s; § 92

and since two ∠s of the △ are equal to the ∠s a and b,

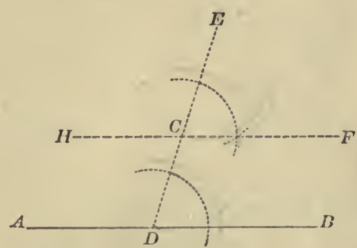
the third ∠ of the △ will be equal to the ∠ c. Ax. 3.

Q. E. F.

Ex. 98. In a triangle ABC, given angles A and B, equal respectively to 37° 13′ 32″ and 41° 17′ 56″. Find the value of angle C.

Proposition XXVIII. Problem.

278. *Through a given point, to draw a straight line parallel to a given straight line.*



Let AB be the given line, and C the given point.

To draw through the point C a line parallel to the line AB.

Construction. Draw DCE, making the $\angle\ EDB$.

At the point C construct $\angle\ ECF = \angle\ EDB$. § 276

Then the line FCH is $\parallel$ to AB.

Proof. $\angle\ ECF = \angle\ EDB$. Cons.

$\therefore\ HF$ is $\parallel$ to AB, § 108

(*when two straight lines, lying in the same plane, are cut by a third straight line, if the ext.-int. ∠s are equal, the lines are parallel*).

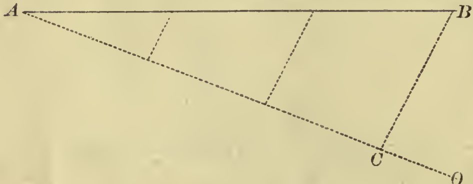
Q. E. F.

Ex. 99. To find a point X equidistant from two given points and also equidistant from two given parallel lines.

Ex. 100. To find a point X equidistant from two given intersecting lines and also equidistant from two given parallels.

Proposition XXIX. Problem.

279. *To divide a given straight line into equal parts.*



Let AB be the given straight line.

To divide AB into equal parts.

Construction. From A draw the line AO.

Take any convenient length, and apply it to AO as many times as the line AB is to be divided into parts.

From the last point thus found on AO, as C, draw CB.

Through the several points of division on AO draw lines ∥ to CB, and these lines divide AB into equal parts.

Proof. Since AC is divided into equal parts, AB is also, § 187

(*if three or more* ∥s *intercept equal parts on any transversal, they intercept equal parts on every transversal*).

Q. E. F.

Ex. 101. To divide a line into four equal parts by two different methods.

Ex. 102. To find a point X in one side of a given triangle and equidistant from the other two sides.

Ex. 103. Through a given point to draw a line which shall make equal angles with the two sides of a given angle.

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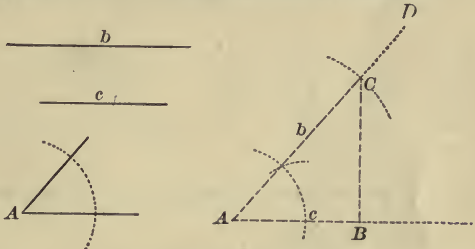
Proposition XXX. Problem.

280. *Two sides and the included angle of a triangle being given, to construct the triangle.*



Let the two sides of the triangle be b and c, and the included angle A.

To construct a △ having two sides equal to b and c respectively, and the included ∠ = ∠ A.

Construction. Take AB equal to the side c.

At A, the extremity of AB, construct an angle equal to the given ∠ A. § 276

On AD take AC equal to b.

Draw CB.

Then △ ACB is the △ required.

Q. E. F.

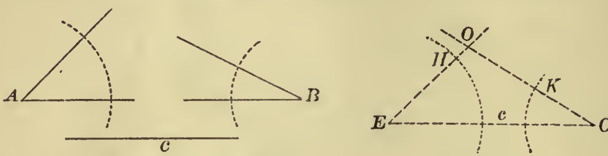
Ex. 104. To construct an angle of 45°.

Ex. 105. To find a point X which shall be equidistant from two given intersecting lines and at a given distance from a given point.

Ex. 106. To draw through two sides of a triangle a line ∥ to the third side so that the part intercepted between the sides shall have a given length.

Proposition XXXI. Problem.

281. *A side and two angles of a triangle being given, to construct the triangle.*



Let c be the given side, A and B the given angles.

To construct the triangle.

Construction. Take EC equal to c.

At the point E construct the $\angle CEH$ equal to $\angle A$. § 276

At the point C construct the $\angle ECK$ equal to $\angle B$.

Let the sides EH and CK intersect at O.

Then $\triangle COE$ is the $\triangle$ required.

Q. E. F.

Remark. If one of the given angles is opposite to the given side, find the third angle by § 277, and proceed as above.

Discussion. The problem is impossible when the two given angles are together equal to or greater than two right angles.

Ex. 107. To construct an angle of 150°.

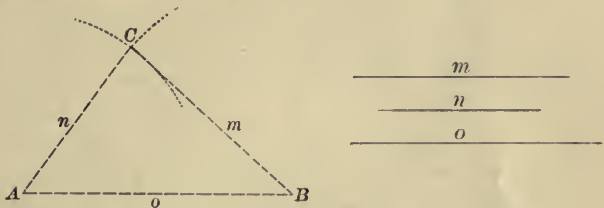
Ex. 108. A straight railway passes two miles from a town. A place is four miles from the town and one mile from the railway. To find by construction how many places answer this description.

Ex. 109. If in a circle two equal chords intersect, the segments of one chord are equal to the segments of the other, each to each.

Ex. 110. AB is any chord and AC is tangent to a circle at A, CDE a line cutting the circumference in D and E and parallel to AB; show that the triangles ACD and EAB are mutually equiangular.

PROPOSITION XXXII. PROBLEM.

282. *The three sides of a triangle being given, to construct the triangle.*



Let the three sides be m, n, and o.

To construct the triangle.

Construction. Draw AB equal to o.

From A as a centre, with a radius equal to n, describe an arc;

and from B as a centre, with a radius equal to m, describe an arc intersecting the former arc at C.

Draw CA and CB.

Then $\triangle CAB$ is the $\triangle$ required.

Q. E. F.

Discussion. The problem is impossible when one side is equal to or greater than the sum of the other two.

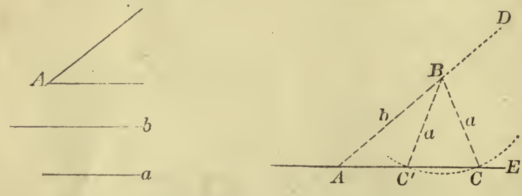
Ex. 111. The base, the altitude, and an angle at the base, of a triangle being given, to construct the triangle.

Ex. 112. Show that the bisectors of the angles contained by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.

Ex. 113. Given two perpendiculars, AB and CD, intersecting in O, and a straight line intersecting these perpendiculars in E and F; to construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie in EF. (Two solutions.)

Proposition XXXIII. Problem.

283. *Two sides of a triangle and the angle opposite one of them being given, to construct the triangle.*



Case I. *If the side opposite to the given angle is less than the other given side.*

Let b be greater than a, and A the given angle.

To construct the triangle.

Construction. Construct $\angle DAE =$ to the given $\angle A$. § 276

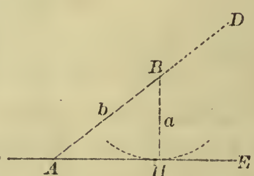
On AD take $AB = b$.

From B as a centre, with a radius equal to a,

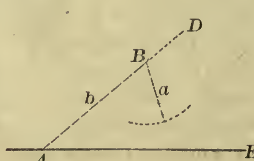
describe an arc intersecting the side AE at C and C'.

Draw BC and BC'.

Then both the $\triangle$ ABC and ABC' fulfil the conditions, and hence we have two constructions. This is called the *ambiguous* case.



Discussion. If the side a is equal to the $\perp$ BH, the arc described from B will touch AE, and there will be but one construction, the right triangle ABH.

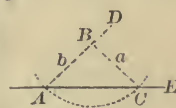


If the given side a is less than the $\perp$ from B, the arc described from B will not intersect or touch AE, and hence the problem is impossible.

If the $\angle A$ is right or obtuse, the problem is impossible; for the side opposite a right or obtuse angle is the greatest side. § 159

Case II. *If a is equal to b.*

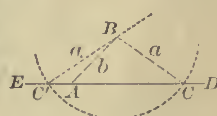
If the $\angle A$ is acute, and $a = b$, the arc described from B as a centre, and with a radius equal to a, will cut the line AE at the points A and C. There is therefore but one solution: the isosceles $\triangle ABC$.



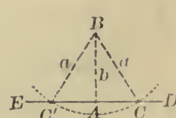
Discussion. If the $\angle A$ is right or obtuse, the problem is impossible; for equal sides of a $\triangle$ have equal $\angle$s opposite them, and a $\triangle$ cannot have two right $\angle$s or two obtuse $\angle$s.

Case III. *If a is greater than b.*

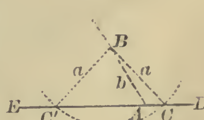
If the given $\angle A$ is acute, the arc described from B will cut the line ED on opposite sides of A, at C and C'. The $\triangle ABC$ answers the required conditions, but the $\triangle ABC'$ does not, for it does not contain the acute $\angle A$. There is then only one solution; namely, the $\triangle ABC$.



If the $\angle A$ is right, the arc described from B cuts the line ED on opposite sides of A, and we have two *equal* right $\triangle$s which fulfil the required conditions.



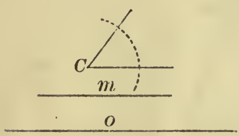
If the $\angle A$ is obtuse, the arc described from B cuts the line ED on opposite sides of A, at the points C and C'. The $\triangle ABC$ answers the required conditions, but the $\triangle ABC'$ does not, for it does not contain the obtuse $\angle A$. There is then only one solution; namely, the $\triangle ABC$.

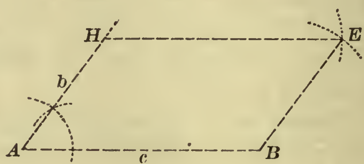


Q. E. F

Proposition XXXIV. Problem.

284. *Two sides and an included angle of a parallelogram being given, to construct the parallelogram.*





Let m and o be the two sides, and C the included angle.

To construct a parallelogram.

Construction. Draw AB equal to o.

At A construct the $\angle A$ equal to $\angle C$, § 276

and take AH equal to m.

From H as a centre, with a radius equal to o, describe an arc.

From B as a centre, with a radius equal to m,

describe an arc, intersecting the former arc at E.

Draw EH and EB.

The quadrilateral $ABEH$ is the ▱ required.

Proof. $AB = HE$, Cons.

$AH = BE$. Cons.

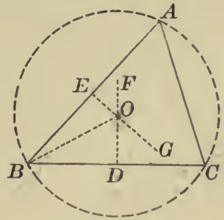
$\therefore$ the figure $ABEH$ is a ▱, § 183

(having its opposite sides equal).

Q. E. F.

Proposition XXXV. Problem.

285. *To circumscribe a circle about a given triangle.*



Let ABC be the given triangle.

To circumscribe a circle about ABC.

Construction. Bisect AB and BC. § 273

At the points of bisection erect ⊥s. § 271

Since BC is not the prolongation of AB, these ⊥s will intersect at some point O.

From O, with a radius equal to OB, describe a circle.

⊙ ABC is the ○ required.

Proof. The point O is equidistant from A and B,

and also is equidistant from B and C, § 122

(*every point in the ⊥ erected at the middle of a straight line is equidistant from the extremities of that line*).

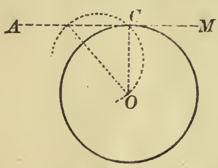
∴ the point O is equidistant from A, B, and C,

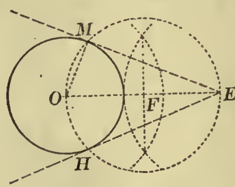
and a ⊙ described from O as a centre, with a radius equal to OB, will pass through the vertices A, B, and C. Q. E. F.

286. Scholium. The same construction serves to describe a circumference which shall pass through the three points not in the same straight line; also to find the centre of a given circle or of a given arc.

Proposition XXXVI. Problem.

287. *Through a given point, to draw a tangent to a given circle.*





Case I. *When the given point is on the circle.*

Let C be the given point on the circle.

To draw a tangent to the circle at C.

Construction. From the centre O draw the radius OC.

Through C draw $AM \perp$ to OC. § 271

Then AM is the tangent required.

Proof. A straight line $\perp$ to a radius at its extremity is tangent to the circle. § 239

Case II. *When the given point is without the circle.*

Let O be the centre of the given circle, E the given point without the circle.

To draw a tangent to the given circle from the point E.

Construction. Join OE.

On OE as a diameter, describe a circumference intersecting the given circumference at the points M and H.

Draw OM and EM.

Then EM is the tangent required.

Proof. $\angle OME$ is a right angle, § 264

(being inscribed in a semicircle).

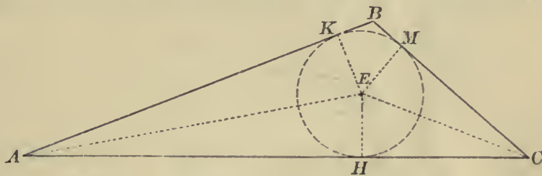
$\therefore EM$ is tangent to the circle at M. § 239

In like manner, we may prove HE tangent to the given $\odot$.

Q. E. F.

Proposition XXXVII. Problem.

288. *To inscribe a circle in a given triangle.*



Let ABC be the given triangle.

To inscribe a circle in the △ *ABC.*

Construction. Bisect ∡ A and C. § 275

From E, the intersection of these bisectors, draw $EH \perp$ to the line AC. § 272

From E, with radius EH, describe the ⊙ KMH.

The ⊙ KHM is the ⊙ required.

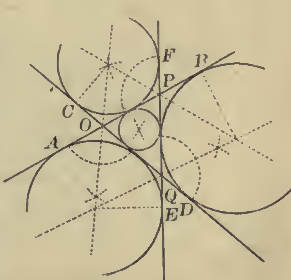
Proof. Since E is in the bisector of the ∠ A, it is equidistant from the sides AB and AC; and since E is in the bisector of the ∠ C, it is equidistant from the sides AC and BC, § 162

(*every point in the bisector of an ∠ is equidistant from the sides of the ∠*).

∴ a ⊙ described from E as centre, with a radius equal to EH, will touch the sides of the △ and be inscribed in it.

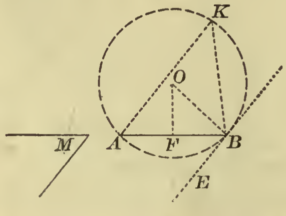
Q. E. F.

289. Scholium. The intersections of the bisectors of exterior angles of a triangle, formed by producing the sides of the triangle, are the centres of three circles, each of which will touch one side of the triangle, and the two other sides produced. These three circles are called *escribed* circles.



Proposition XXXVIII. Problem.

290. *Upon a given straight line, to describe a segment of a circle which shall contain a given angle.*



Let AB be the given line, and M the given angle.

To describe a segment upon AB which shall contain $\angle M$.

Construction. Construct $\angle ABE$ equal to $\angle M$. § 276

Bisect the line AB by the $\perp$ FO. § 273

From the point B draw $BO \perp$ to EB. § 271

From O, the point of intersection of FO and BO, as a centre, with a radius equal to OB, describe a circumference.

The segment AKB is the segment required.

Proof. The point O is equidistant from A and B, § 122

(*every point in a $\perp$ erected at the middle of a straight line is equidistant from the extremities of that line*).

$\therefore$ the circumference will pass through A.

But BE is $\perp$ to OB. Cons.

$\therefore$ BE is tangent to the $\odot$, § 239

(*a straight line $\perp$ to a radius at its extremity is tangent to the $\odot$*).

$\therefore$ $\angle ABE$ is measured by $\frac{1}{2}$ arc AB, § 269

(*being an $\angle$ formed by a tangent and a chord*).

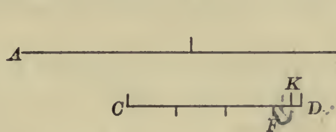
An $\angle$ inscribed in the segment AKB is measured by $\frac{1}{2}$ AB. § 263

$\therefore$ segment AKB contains $\angle M$. Ax. 1

Q. E. F.

PROPOSITION XXXIX. PROBLEM.

291. *To find the ratio of two commensurable straight lines.*



Let AB and CD be two straight lines.

To find the ratio of AB and CD.

Apply CD to AB as many times as possible.

Suppose twice, with a remainder EB.

Then apply EB to CD as many times as possible.

Suppose three times, with a remainder FD.

Then apply FD to EB as many times as possible.

Suppose once, with a remainder HB.

Then apply HB to FD as many times as possible.

Suppose once, with a remainder KD.

Then apply KD to HB as many times as possible.

Suppose KD is contained just twice in HB.

The measure of each line, referred to KD as a unit, will then be as follows:

$$HB = 2\,KD;$$
$$FD = HB + KD = 3\,KD;$$
$$EB = FD + HB = 5\,KD;$$
$$CD = 3\,EB + FD = 18\,KD;$$
$$AB = 2\,CD + EB = 41\,KD;$$
$$\therefore \frac{AB}{CD} = \frac{41\,KD}{18\,KD};$$
$$\therefore \text{the ratio } \frac{AB}{CD} = \frac{41}{18}.$$

Q. E. F.

Theorems.

114. The shortest line and the longest line which can be drawn from a given point to a given circumference pass through the centre.

115. The shortest chord that can be drawn through a given point within a given circle is $\perp$ to the diameter which passes through the point.

116. In the same circle, or in equal circles, if two arcs are each greater than a semi-circumference, the greater arc subtends the *less* chord, and conversely.

117. If ABC is an inscribed equilateral triangle, and P is any point in the arc BC, then $PA = PB + PC$.

Hint. On PA take PM equal to PB, and join BM.

118. In what kinds of parallelograms can a circle be inscribed? Prove your answer.

119. The radius of the circle inscribed in an equilateral triangle is equal to one-third of the altitude of the triangle.

120. A circle can be circumscribed about a rectangle.

121. A circle can be circumscribed about an isosceles trapezoid.

122. The tangents drawn through the vertices of an inscribed rectangle enclose a rhombus.

123. The diameter of the circle inscribed in a rt. $\triangle$ is equal to the difference between the sum of the legs and the hypotenuse.

124. From a point A without a circle, a diameter AOB is drawn, and also a secant ACD, so that the part AC without the circle is equal to the radius. Prove that the $\angle DAB$ equals one-third the $\angle DOB$.

125. All chords of a circle which touch an interior concentric circle are equal, and are bisected at the points of contact.

126. If two circles intersect, and a secant is drawn through each point of intersection, the chords which join the extremities of the secants are parallel. Hint. By drawing the common chord, two inscribed quadrilaterals are obtained.

127. If an equilateral triangle is inscribed in a circle, the distance of each side from the centre of the circle is equal to half the radius of the circle.

128. Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the straight line joining the ends of the diameters passes through the other point of intersection.

129. A circle touches two sides of an angle BAC at B, C; through any point D in the arc BC a tangent is drawn, meeting AB at E and AC at F. Prove (i.) that the perimeter of the triangle AEF is constant for all positions of D in BC; (ii.) that the angle EOF is also constant.

Loci.

130. Find the locus of a point at three inches from a given point.

131. Find the locus of a point at a given distance from a given circumference.

132. Prove that the locus of the vertex of a right triangle, having a given hypotenuse as base, is the circumference described upon the given hypotenuse as diameter.

133. Prove that the locus of the vertex of a triangle, having a given base and a given angle at the vertex, is the arc which forms with the base a segment capable of containing the given angle.

134. Find the locus of the middle points of all chords of a given length that can be drawn in a given circle.

135. Find the locus of the middle points of all chords that can be drawn through a given point A in a given circumference.

136. Find the locus of the middle points of all secants that can be drawn from a given point A to a given circumference.

137. A straight line moves so that it remains parallel to a given line, and touches at one end a given circumference. Find the locus of the other end.

138. A straight rod moves so that its ends constantly touch two fixed rods which are $\perp$ to each other. Find the locus of its middle point.

139. In a given circle let AOB be a diameter, OC any radius, CD the perpendicular from C to AB. Upon OC take $OM = CD$. Find the locus of the point M as OC turns about O.

Construction of Polygons.

To construct an equilateral $\triangle$, having given:

140. The perimeter.

141. The radius of the circumscribed circle.

142. The altitude.

143. The radius of the inscribed circle.

To construct an isosceles triangle, having given:

144. The angle at the vertex and the base.

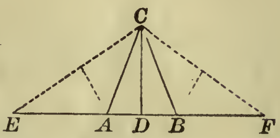
145. The angle at the vertex and the altitude.

146. The base and the radius of the circumscribed circle.

147. The base and the radius of the inscribed circle.

148. The perimeter and the altitude.

HINTS. Let ABC be the $\triangle$ required, and EF the given perimeter. The altitude CD passes through the middle of EF, and the $\triangle$s AEC, BFC are isosceles.



To construct a right triangle, having given:

149. The hypotenuse and one leg.

150. The hypotenuse and the altitude upon the hypotenuse.

151. One leg and the altitude upon the hypotenuse as base.

152. The median and the altitude drawn from the vertex of the rt. $\angle$.

153. The radius of the inscribed circle and one leg.

154. The radius of the inscribed circle and an acute angle.

155. An acute angle and the sum of the legs.

156. An acute angle and the difference of the legs.

To construct a triangle, having given:

157. The base, the altitude, and the $\angle$ at the vertex.

158. The base, the corresponding median, and the $\angle$ at the vertex.

159. The perimeter and the angles.

160. One side, an adjacent $\angle$, and the sum of the other sides.

161. One side, an adjacent $\angle$, and the difference of the other sides.

162. The sum of two sides and the angles.

163. One side, an adjacent $\angle$, and radius of circumscribed $\odot$.

164. The angles and the radius of the circumscribed $\odot$.

165. The angles and the radius of the inscribed $\odot$.

166. An angle, the bisector, and the altitude drawn from the vertex

167. Two sides and the median corresponding to the other side.

168. The three medians.

To construct a square, having given:

169. The diagonal. 170. The sum of the diagonal and one side.

To construct a rectangle, having given:

171. One side and the ∠ formed by the diagonals.

172. The perimeter and the diagonal.

173. The perimeter and the ∠ of the diagonals.

174. The difference of the two adjacent sides and the ∠ of the diagonals.

To construct a rhombus, having given:

175. The two diagonals.

176. One side and the radius of the inscribed circle.

177. One angle and the radius of the inscribed circle.

178. One angle and one of the diagonals.

To construct a rhomboid, having given:

179. One side and the two diagonals.

180. The diagonals and the ∠ formed by them.

181. One side, one ∠, and one diagonal.

182. The base, the altitude, and one angle.

To construct an isosceles trapezoid, having given:

183. The bases and one angle. 184. The bases and the altitude.

185. The bases and the diagonal.

186. The bases and the radius of the circumscribed circle.

To construct a trapezoid, having given:

187. The four sides. 188. The two bases and the two diagonals.

189. The bases, one diagonal, and the ∠ formed by the diagonals.

Construction of Circles.

Find the locus of the centre of a circle:

190. Which has a given radius r and passes through a given point P.

191. Which has a given radius r and touches a given straight line AB

192. Which passes through two given points P and Q.

193. Which touches a given straight line AB at a given point P.

194. Which touches each of two given parallels.

195. Which touches each of two given intersecting lines.

To construct a circle which has the radius r and which also:

196. Touches each of two intersecting lines AB and CD.

197. Touches a given line AB and a given circle K.

198. Passes through a given point P and touches a given line AB.

199. Passes through a given point P and touches a given circle K.

To construct a circle which shall:

200. Touch two given parallels and pass through a given point P.

201. Touch three given lines two of which are parallel.

202. Touch a given line AB at P and pass through a given point Q.

203. Touch a given circle at P and pass through a given point Q.

204. Touch two given lines and touch one of them at a given point P.

205. Touch a given line and touch a given circle at a point P.

206. Touch a given line AB at P and also touch a given circle.

207. To inscribe a circle in a given sector.

208. To construct within a given circle three equal circles, so that each shall touch the other two and also the given circle.

209. To describe circles about the vertices of a given triangle as centres, so that each shall touch the two others.

Construction of Straight Lines.

210. To draw a common tangent to two given circles.

211. To bisect the angle formed by two lines, without producing the lines to their point of intersection.

212. To draw a line through a given point, so that it shall form with the sides of a given angle an isosceles triangle.

213. Given a point P between the sides of an angle BAC. To draw through P a line terminated by the sides of the angle and bisected at P.

214. Given two points P, Q, and a line AB; to draw lines from P and Q which shall meet on AB and make equal angles with AB.

Hint. Make use of the point which forms with P a pair of points symmetrical with respect to AB.

215. To find the shortest path from P to Q which shall touch a line AB.

216. To draw a tangent to a given circle, so that it shall be parallel to a given straight line.

Review to p 159 inc

BOOK III.

PROPORTIONAL LINES AND SIMILAR POLYGONS.

THE THEORY OF PROPORTION.

292. A *proportion* is an expression of equality between two equal ratios.

A proportion may be expressed in any one of the following forms:

$$\frac{a}{b}=\frac{c}{d}; \quad a:b=c:d; \quad a:b::c:d;$$

and is read, "the ratio of a to b equals the ratio of c to d."

293. The *terms* of a proportion are the four quantities compared; the *first* and *third* terms are called the *antecedents*, the *second* and *fourth* terms, the *consequents*; the *first* and *fourth* terms are called the *extremes*, the *second* and *third* terms, the *means*.

294. In the proportion $a:b=c:d$, d is a *fourth proportional* to a, b, and c.

In the proportion $a:b=b:c$, c is a *third proportional* to a and b.

In the proportion $a:b=b:c$, b is a *mean proportional* between a and c.

Proposition I.

295. *In every proportion the product of the extremes is equal to the product of the means.*

Let $a : b = c \cdot d$.

To prove $ad = bc$.

Now $\frac{a}{b} = \frac{c}{d},$

whence, by multiplying both sides by bd,

$$ad = bc. \qquad \text{Q. E. D.}$$

Proposition II.

296. *A mean proportional between two quantities is equal to the square root of their product.*

In the proportion $a : b = b : c$,

$$b^2 = ac, \qquad \S\ 295$$

(*the product of the extremes is equal to the product of the means*).

Whence, extracting the square root,

$$b = \sqrt{ac}. \qquad \text{Q. E. D.}$$

Proposition III.

297. *If the product of two quantities is equal to the product of two others, either two may be made the extremes of a proportion in which the other two are made the means.*

Let $ad = bc$.

To prove $a : b = c : d$.

Divide both members of the given equation by bd.

Then $\frac{a}{b} = \frac{c}{d},$

or, $a : b = c : d$. Q. E. D.

Proposition IV.

298. *If four quantities of the same kind are in proportion, they will be in proportion by* **alternation**; *that is, the first term will be to the third as the second to the fourth.*

Let $a : b = c : d$.

To prove $a : c = b : d$.

Now $\frac{a}{b} = \frac{c}{d}$.

Multiply each member of the equation by $\frac{b}{c}$.

Then $\frac{a}{c} = \frac{b}{d}$,

or, $a : c = b : d$.

Q. E. D.

Proposition V.

299. *If four quantities are in proportion, they will be in proportion by* **inversion**; *that is, the second term will be to the first as the fourth to the third.*

Let $a : b = c : d$.

To prove $b : a = d : c$.

Now $bc = ad$. § 295

Divide each member of the equation by ac.

Then $\frac{b}{a} = \frac{d}{c}$,

or, $b : a = d : c$.

Q. E. D.

PROPOSITION VI.

300. *If four quantities are in proportion, they will be in proportion by* **composition**; *that is, the sum of the first two terms will be to the second term as the sum of the last two terms to the fourth term.*

Let $a : b = c : d$.

To prove $a + b : b = c + d : d$.

Now $\frac{a}{b} = \frac{c}{d}$.

Add 1 to each member of the equation.

Then $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

that is, $\frac{a+b}{b} = \frac{c+d}{d}$,

or, $a + b : b = c + d : d$.

In like manner, $a + b : a = c + d : c$.

Q. E. D.

PROPOSITION VII.

301. *If four quantities are in proportion, they will be in proportion by* **division**; *that is, the difference of the first two terms will be to the second term as the difference of the last two terms to the fourth term.*

Let $a : b = c : d$.

To prove $a - b : b = c - d : d$.

Now $\frac{a}{b} = \frac{c}{d}$.

Subtract 1 from each member of the equation.

Then $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

that is, $\frac{a-b}{b} = \frac{c-d}{d}$,

or, $a - b : b = c - d : d$.

In like manner, $a - b : a = c - d : c$.

Q. E. D.

Proposition VIII.

302. *In any proportion the terms are in proportion by* **composition and division**; *that is, the sum of the first two terms is to their difference as the sum of the last two terms to their difference.*

Let $a : b = c : d$.

Then, by § 300, $\frac{a+b}{a} = \frac{c+d}{c}$.

And, by § 301, $\frac{a-b}{a} = \frac{c-d}{c}$.

By division, $\frac{a+b}{a-b} = \frac{c+d}{c-d}$,

or, $a+b : a-b = c+d : c-d$.

Q. E. D.

Proposition IX.

303. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let $a : b = c : d = e : f = g : h$.

To prove $a+c+e+g : b+d+f+h = a : b$.

Denote each ratio by r.

Then $r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$.

Whence, $a = br$, $c = dr$, $e = fr$, $g = hr$.

Add these equations.

Then $a+c+e+g = (b+d+f+h)r$.

Divide by $(b+d+f+h)$.

Then $\frac{a+c+e+g}{b+d+f+h} = r = \frac{a}{b}$,

or, $a+c+e+g : b+d+f+h = a : b$.

Q. E. D

Proposition X.

304. *The products of the corresponding terms of two or more proportions are in proportion.*

Let $a:b=c:d,\quad e:f=g:h,\quad k:l=m:n.$

To prove $aek:bfl=cgm:dhn.$

Now $\frac{a}{b}=\frac{c}{d},\quad \frac{e}{f}=\frac{g}{h},\quad \frac{k}{l}=\frac{m}{n}.$

Whence, by multiplication,

$$\frac{aek}{bfl}=\frac{cgm}{dhn},$$

or, $aek:bfl=cgm:dhn.$

Q. E. D.

Proposition XI.

305. *Like powers, or like roots, of the terms of a proportion are in proportion.*

Let $a:b=c:d.$

To prove $a^n:b^n=c^n:d^n,$

and $a^{\frac{1}{n}}:b^{\frac{1}{n}}=c^{\frac{1}{n}}:d^{\frac{1}{n}}.$

Now $\frac{a}{b}=\frac{c}{d}.$

By raising to the *nth* power,

$$\frac{a^n}{b^n}=\frac{c^n}{d^n};\text{ or } a^n:b^n=c^n:d^n.$$

By extracting the *nth* root,

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}};\text{ or, } a^{\frac{1}{n}}:b^{\frac{1}{n}}=c^{\frac{1}{n}}:d^{\frac{1}{n}}.$$

Q. E. D.

306. *Equimultiples* of two quantities are the products obtained by multiplying each of them by the same number. Thus, ma and mb are equimultiples of a and b.

Proposition XII.

307. *Equimultiples of two quantities are in the same ratio as the quantities themselves.*

Let a and b be any two quantities.

To prove $ma : mb = a : b$.

Now $\frac{a}{b} = \frac{a}{b}$.

Multiply both terms of first fraction by m.

Then $\frac{ma}{mb} = \frac{a}{b}$,

or, $ma : mb = a : b$.

Q. E. D.

308. Scholium. In the treatment of proportion it is assumed that fractions may be found which will *represent* the ratios. It is evident that the ratio of two quantities may be represented by a fraction when the two quantities compared can be expressed in *integers* in terms of a *common unit.* But when there is no unit in terms of which both quantities can be expressed in *integers*, it is possible to find a fraction that will represent the ratio to *any required degree of accuracy.* (See §§ 251–256.)

Hence, in speaking of the product of two quantities, as for instance, the product of two lines, we mean simply *the product of the numbers which represent them when referred to a common unit.*

An interpretation of this kind must be given to the product of any two quantities throughout the Geometry.

Proportional Lines.

Proposition I. Theorem.

309. *If a line is drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally.*

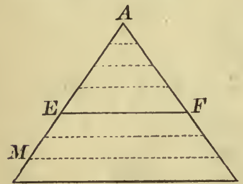


Fig. 1.

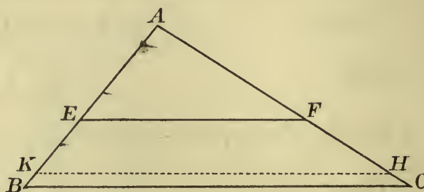


Fig. 2.

***In the triangle** ABC **let** EF **be drawn parallel to** BC.*

To prove $$\frac{EB}{AE}=\frac{FC}{AF}.$$

Case I. *When AE and EB* (Fig. 1) *are commensurable.*

Find a common measure of AE and EB, as BM.

Suppose BM to be contained in BE three times,

and in AE four times.

Then $$\frac{EB}{AE}=\frac{3}{4}. \quad (1)$$

At the several points of division on BE and AE draw straight lines ‖ to BC.

These lines will divide AC into seven equal parts, of which FC will contain three, and AF will contain four, § 187

(*if parallels intercept equal parts on any transversal, they intercept equal parts on every transversal*).

$$\therefore \frac{FC}{AF}=\frac{3}{4}. \quad (2)$$

Compare (1) and (2),

$$\frac{EB}{AE}=\frac{FC}{AF}. \qquad \text{Ax. 1.}$$

Case II. *When AE and EB* (Fig. 2) *are incommensurable.*

Divide AE into any number of equal parts, and apply one of these parts as a unit of measure to EB as many times as it will be contained in EB.

Since AE and EB are incommensurable, a certain number of these parts will extend from E to a point K, leaving a remainder KB less than the unit of measure.

Draw $KH \parallel$ to BC.

Then $$\frac{EK}{AE} = \frac{FH}{AF}. \qquad \text{Case I.}$$

Suppose the unit of measure indefinitely diminished, the ratios $\frac{EK}{AE}$ and $\frac{FH}{AF}$ continue equal; and approach indefinitely the limiting ratios $\frac{EB}{AE}$ and $\frac{FC}{AF}$, respectively.

Therefore $$\frac{EB}{AE} = \frac{FC}{AF}. \qquad \S 260$$

Q. E. D.

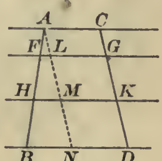
310. Cor. 1. *One side of a triangle is to either part cut off by a straight line parallel to the base as the other side is to the corresponding part.*

For $EB : AE = FC : AF$, by the theorem.

$$\therefore EB + AE : AE = FC + AF : AF, \qquad \S 300$$

or $$AB : AE = AC : AF.$$

311. Cor. 2. *If two lines are cut by any number of parallels, the corresponding intercepts are proportional.*



Let the lines be AB and CD.

Draw $AN \parallel$ to CD, cutting the $\parallel$s at L, M, and N. Then

$AL = CG$, $LM = GK$, $MN = KD$. §180

By the theorem,

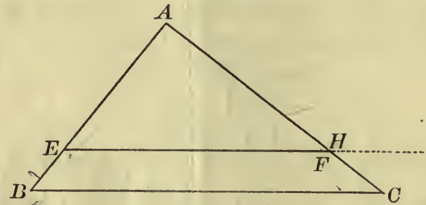
$$AH : AM = AF : AL = FH : LM = HB : MN.$$

That is, $AF : CG = FH : GK = HB : KD$.

If the two lines AB and CD were parallel, the corresponding intercepts would be equal, and the above proportion be true.

PROPOSITION II. THEOREM.

312. *If a straight line divide two sides of a triangle proportionally, it is parallel to the third side.*



In the triangle ABC let EF be drawn so that

$$\frac{AB}{AE} = \frac{AC}{AF}.$$

To prove $EF \parallel$ *to* BC.

Proof. From E draw $EH \parallel$ to BC.

Then $AB : AE = AC : AH$, § 310

(*one side of a* $\triangle$ *is to either part cut off by a line* $\parallel$ *to the base, as the other side is to the corresponding part*).

But $AB : AE = AC : AF$. Hyp.

The last two proportions have the first three terms equal, each to each; therefore the fourth terms are equal; that is,

$$AF = AH.$$

$\therefore$ EF and EH coincide.

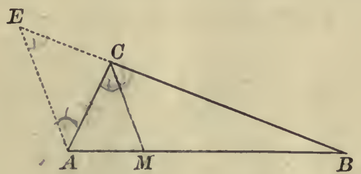
But EH is $\parallel$ to BC. Cons.

$\therefore$ EF, which coincides with EH, is $\parallel$ to BC.

Q. E. D

Proposition III. Theorem.

313. *The bisector of an angle of a triangle divides the opposite side into segments proportional to the other two sides.*



Let CM bisect the angle C of the triangle CAB.

To prove $MA : MB = CA : CB.$

Proof. Draw $AE \parallel$ to CM to meet BC produced at E.

Since CM is $\parallel$ to AE of the $\triangle\ BAE$, we have § 309

$$MA : MB = CE : CB. \qquad (1)$$

Since CM is $\parallel$ to AE,

$$\angle ACM = \angle CAE, \qquad \S\ 104$$

(being alt.-int. ∠s of ∥ lines);

and $$\angle BCM = \angle CEA, \qquad \S\ 106$$

(being ext.-int. ∠s of ∥ lines).

But the $\angle ACM = \angle BCM.$ Hyp.

$$\therefore \text{the } \angle CAE = \angle CEA. \qquad \text{Ax. 1}$$

$$\therefore CE = CA, \qquad \S\ 156$$

(if two ∠s of a △ are equal, the opposite sides are equal).

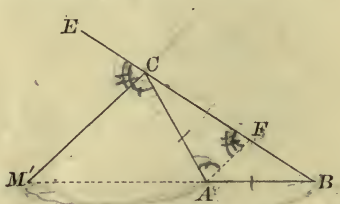
Putting CA for CE in (1), we have

$$MA : MB = CA : CB.$$

Q. E. D.

PROPOSITION IV. THEOREM.

314. *The bisector of an exterior angle of a triangle meets the opposite side produced at a point the distances of which from the extremities of this side are proportional to the other two sides.*



Let CM' bisect the exterior angle ACE of the triangle CAB, and meet BA produced at M'.

To prove $M'A : M'B = CA : CB$.

Proof. Draw $AF \parallel$ to CM' to meet BC at F.

Since AF is $\parallel$ to CM' of the $\triangle$ BCM', we have § 309

$$M'A : M'B = CF : CB. \qquad (1)$$

Since AF is $\parallel$ to CM',

the $\angle M'CE = \angle AFC$, § 106

(being ext.-int. $\angle$s *of* $\parallel$ *lines)*;

and the $\angle M'CA = \angle CAF$, § 104

(being alt.-int. $\angle$s *of* $\parallel$ *lines)*.

Since CM' bisects the $\angle ECA$,

$$\angle M'CE = \angle M'CA.$$

$\therefore$ the $\angle AFC = \angle CAF$. Ax. 1

$\therefore CA = CF$, § 156

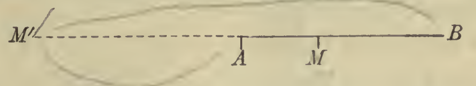
(if two $\angle$s *of a* $\triangle$ *are equal, the opposite sides are equal)*.

Putting CA for CF in (1), we have

$$M'A : M'B = CA : CB.$$

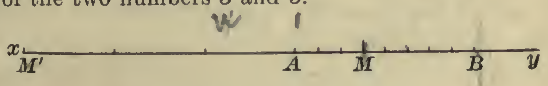
Q. E. D.

315. SCHOLIUM. If a given line AB is divided at M, a point between the extremities A and B, it is said to be divided *internally* into the segments MA and MB; and if it is divided at M', a point in the prolongation of AB, it is said to be divided *externally* into the segments $M'A$ and $M'B$.



In either case the segments are the *distances* from the point of division to the extremities of the line. If the line is divided internally, the *sum* of the segments is equal to the line; and if the line is divided externally, the *difference* of the segments is equal to the line.

Suppose it is required to divide the given line AB internally and externally in the *same ratio*; as, for example, the ratio of the two numbers 3 and 5.



We divide AB into $5 + 3$, or 8, equal parts, and take 3 parts from A; we then have the point M, such that

$$MA : MB = 3 : 5. \tag{1}$$

Secondly, we divide AB into two equal parts, and lay off on the prolongation of AB, to the left of A, three of these equal parts; we then have the point M', such that

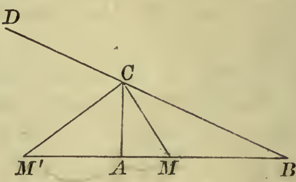
$$M'A : M'B = 3 : 5. \tag{2}$$

Comparing (1) and (2),

$$MA : MB = M'A : M'B.$$

316. If a given straight line is divided internally and externally into segments having the same ratio, the line is said to be *divided harmonically*.

317. Cor. 1. *The bisectors of an interior angle and an exterior angle at one vertex of a triangle divide the opposite side harmonically.* For, by §§ 313 and 314, each bisector divides the opposite side into segments proportional to the other two sides of the triangle.



318. Cor. 2. *If the points M and M' divide the line AB harmonically, the points A and B divide the line MM' harmonically.*

For, if $MA : MB = M'A : M'B$,

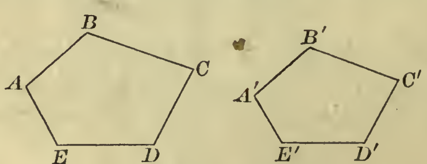
by alternation, $MA : M'A = MB : M'B$. § 298

That is, the ratio of the distances of A from M and M' is equal to the ratio of the distances of B from M and M'.

The four points A, B, M, and M' are called *harmonic points*, and the two pairs, A, B, and M, M', are called *conjugate harmonic points.*

Similar Polygons.

319. *Similar* polygons are polygons that have their homologous angles equal, and their homologous sides proportional.



Thus, if the polygons $ABCDE$ and $A'B'C'D'E'$ are similar

the $\angle$s A, B, C, etc., are equal to $\angle$s A', B', C', etc.

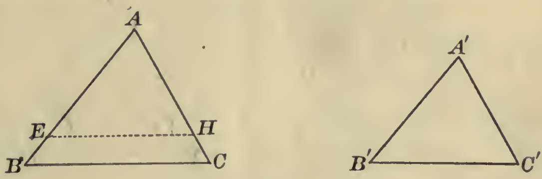
and $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

320. In two similar polygons, the ratio of any two homologous sides is called the *ratio of similitude* of the polygons.

SIMILAR TRIANGLES.

PROPOSITION V. THEOREM.

321. *Two mutually equiangular triangles are similar.*



In the triangles ABC and A′B′C′ let angles A, B, C be equal to angles A′, B′, C′ respectively.

To prove $\triangle$s *ABC and A′B′C′ similar.*

Proof. Apply the $\triangle\ A'B'C'$ to the $\triangle\ ABC$,

so that $\angle\ A'$ shall coincide with $\angle\ A$.

Then the $\triangle\ A'B'C'$ will take the position of $\triangle\ AEH$.

Now $\angle\ AEH$ (same as $\angle\ B'$) $= \angle\ B$.

$\therefore\ EH$ is $\parallel$ to BC, § 108

(*when two straight lines, lying in the same plane, are cut by a third straight line, if the ext.-int.* $\angle$s *are equal the lines are parallel*).

$\therefore\ AB : AE = AC : AH$, § 310

or $AB : A'B' = AC : A'C'$.

In like manner, by applying $\triangle\ A'B'C'$ to $\triangle\ ABC$, so that $\angle\ B'$ shall coincide with $\angle\ B$, we may prove that

$$AB : A'B' = BC : B'C'.$$

Therefore the two $\triangle$s are similar. § 319

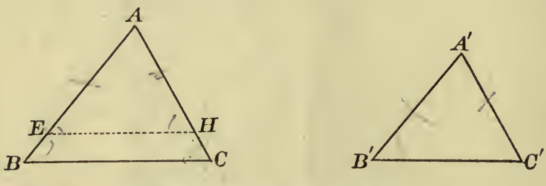
Q. E. D.

322. COR. 1. *Two triangles are similar if two angles of the one are equal respectively to two angles of the other.*

323. COR. 2. *Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.*

PROPOSITION VI. THEOREM.

324. *If two triangles have their sides respectively proportional, they are similar.*



In the triangles ABC and A′B′C′ let

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

To prove $\triangle s$ *ABC and A′B′C′ similar.*

Proof. Take $AE = A'B'$, and $AH = A'C'$.

Draw EH.

Then from the given proportion,

$$\frac{AB}{AE} = \frac{AC}{AH}.$$

$\therefore$ EH is $\parallel$ to BC, § 312

(*if a line divide two sides of a $\triangle$ proportionally, it is $\parallel$ to the third side*).

Hence in the $\triangle s$ ABC and AEH

$\angle ABC = \angle AEH$, § 106

and $\angle ACB = \angle AHE$,

(*being ext.-int. $\angle s$ of $\parallel$ lines*).

$\therefore$ $\triangle s$ ABC and AEH are similar, § 322

(*two $\triangle s$ are similar if two $\angle s$ of one are equal respectively to two $\angle s$ of the other*).

$\therefore AB : AE = BC : EH;$

that is, $AB : A'B' = BC : EH.$

But by hypothesis,

$$AB : A'B' = BC : B'C'.$$

The last two proportions have the first three terms equal each to each; therefore the fourth terms are equal; that is,

$$EH = B'C'.$$

Hence in the $\triangle$ AEH and $A'B'C'$,

$$EH = B'C', \ AE = A'B', \text{ and } AH = A'C'.$$

$$\therefore \triangle AEH = \triangle A'B'C', \qquad \S\ 160$$

(*having three sides of the one equal respectively to three sides of the other*).

But $\triangle AEH$ is similar to $\triangle ABC$.

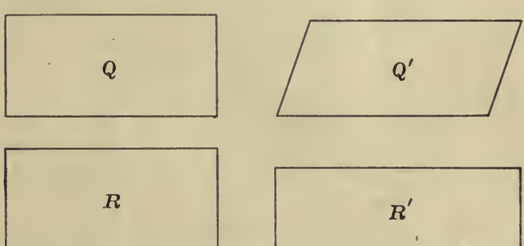
$\therefore \triangle A'B'C'$ is similar to $\triangle ABC$. Q. E. D.

325. Scholium. The primary idea of similarity is *likeness of form;* and the two conditions necessary to similarity are:

I. For every angle in one of the figures there must be an equal angle in the other, and

II. The homologous sides must be in proportion.

In the case of *triangles*, either condition involves the other, but in the case of *other polygons*, it does not follow that if one condition exist the other does also.

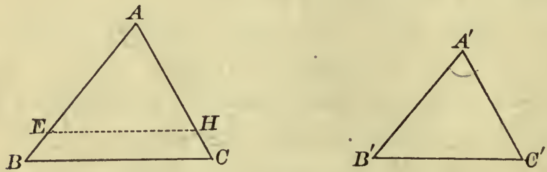


Thus in the quadrilaterals Q and Q', the homologous sides are proportional, but the homologous angles are not equal.

In the quadrilaterals R and R' the homologous angles are equal, but the sides are not proportional.

Proposition VII. Theorem.

326. *If two triangles have an angle of the one equal to an angle of the other, and the including sides proportional, they are similar.*



In the triangles** ABC **and** $A'B'C'$, **let** $\angle A = \angle A'$, **and

$$\frac{AB}{A'B'} = \frac{AC}{A'C'}.$$

To prove $\triangle$s ABC *and* $A'B'C'$ *similar.*

Proof. Apply the $\triangle$ $A'B'C'$ to the $\triangle$ ABC, so that $\angle$ A' shall coincide with $\angle$ A.

Then the $\triangle$ $A'B'C'$ will take the position of $\triangle$ AEH.

Now $$\frac{AB}{A'B'} = \frac{AC}{A'C'}.$$ Hyp.

That is, $$\frac{AB}{AE} = \frac{AC}{AH}.$$

Therefore the line EH divides the sides AB and AC proportionally;

$\therefore$ EH is $\parallel$ to BC, § 312

(*if a line divide two sides of a* $\triangle$ *proportionally, it is* $\parallel$ *to the third side*).

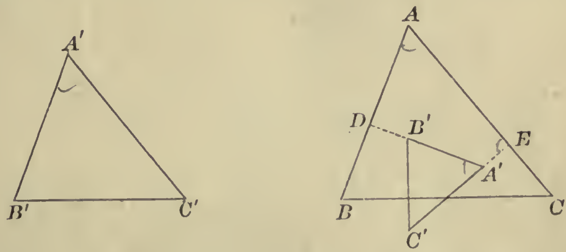
Hence the $\triangle$s ABC and AEH are mutually equiangular and similar.

$\therefore$ $\triangle$ $A'B'C'$ is similar to $\triangle$ ABC.

Q. E. D.

PROPOSITION VIII. THEOREM.

327. *If two triangles have their sides respectively parallel, or respectively perpendicular, they are similar.*



In the triangles $A'B'C'$ and ABC let $A'B'$, $A'C'$, $B'C'$ be respectively parallel, or respectively perpendicular, to AB, AC, BC.

To prove $\triangle$ *$A'B'C'$ and ABC similar.*

Proof. The corresponding $\angle$s are either equal or supplements of each other, §§ 112, 113

(*if two $\angle$s have their sides* ||, *or* $\perp$, *they are equal or supplementary*).

Hence we may make three suppositions:

1st. $A + A' = 2$ rt. $\angle$s, $B + B' = 2$ rt. $\angle$s, $C + C' = 2$ rt. $\angle$s.
2d. $A = A'$, $B + B' = 2$ rt. $\angle$s, $C + C' = 2$ rt. $\angle$s.
3d. $A = A'$, $B = B'$, $\therefore C = C'$. § 140

Since the sum of the $\angle$s of the two $\triangle$ cannot exceed *four right angles*, the third supposition only is admissible. § 138

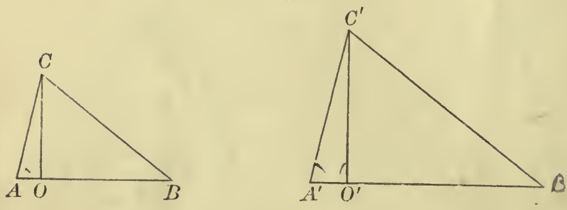
$\therefore$ the two $\triangle$ ABC and $A'B'C'$ are similar, § 321

(*two mutually equiangular* $\triangle$ *are similar*).

Q. E. D

Proposition IX. Theorem.

328. *The homologous altitudes of two similar triangles have the same ratio as any two homologous sides.*



***In the two similar triangles** ABC **and** $A'B'C'$**, let the altitudes be** CO **and** $C'O'$.*

To prove $$\frac{CO}{C'O'} = \frac{AC}{A'C'} = \frac{AB}{A'B'}.$$

Proof. In the rt. △ COA and $C'O'A'$,

$$\angle A = \angle A', \qquad \text{§ 319}$$

(*being homologous ∠ of the similar △ ABC and $A'B'C'$*).

$\therefore$ △ COA and $C'O'A'$ are similar, § 323

(*two rt. △ having an acute ∠ of the one equal to an acute ∠ of the other are similar*).

$$\therefore \frac{CO}{C'O'} = \frac{AC}{A'C'}. \qquad \text{§ 319}$$

In the similar △ ABC and $A'B'C'$,

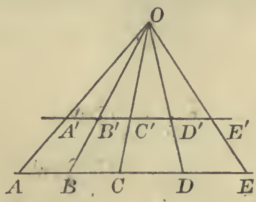
$$\frac{AC}{A'C'} = \frac{AB}{A'B'}.$$

Therefore, $$\frac{CO}{C'O'} = \frac{AC}{A'C'} = \frac{AB}{A'B'}.$$

Q. E. D.

PROPOSITION X. THEOREM.

329. *Straight lines drawn through the same point intercept proportional segments upon two parallels.*



Let the two parallels AE and A'E' cut the straight lines OA, OB, OC, OD, and OE.

To prove $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\frac{DE}{D'E'}$.

Proof. Since $A'E'$ is ∥ to AE, the pairs of △ OAB and $OA'B'$, OBC and $OB'C'$, etc., are mutually equiangular and similar,

$$\therefore \frac{AB}{A'B'}=\frac{OB}{OB'} \text{ and } \frac{BC}{B'C'}=\frac{OB}{OB'}, \qquad \S\ 319$$

(*homologous sides of similar △ are proportional*).

$$\therefore \frac{AB}{A'B'}=\frac{BC}{B'C'}. \qquad \text{Ax. 1}$$

In a similar way it may be shown that

$$\frac{BC}{B'C'}=\frac{CD}{C'D'} \text{ and } \frac{CD}{C'D'}=\frac{DE}{D'E'}.$$

Q. E. D

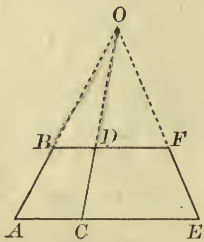
REMARK. A condensed form of writing the above is

$$\frac{AB}{A'B'}=\left(\frac{OB}{OB'}\right)=\frac{BC}{B'C'}=\left(\frac{OC}{OC'}\right)=\frac{CD}{C'D'}=\left(\frac{OD}{OD'}\right)=\frac{DE}{D'E'},$$

where a parenthesis about a ratio signifies that this ratio is used to prove the equality of the ratios immediately preceding and following it.

Proposition XI. Theorem.

330. Conversely: *If three or more non-parallel straight lines intercept proportional segments upon two parallels, they pass through a common point.*



Let AB, CD, EF, cut the parallels AE and BF so that $AC : BD = CE : DF$.

To prove that AB, CD, EF prolonged meet in a point.

Proof. Prolong AB and CD until they meet in O.

Join OE.

If we designate by F' the point where OE cuts BF, we shall have by § 329,

$$AC : BD = CE : DF'.$$

But by hypothesis

$$AC : BD = CE : DF.$$

These proportions have the first three terms equal, each to each; therefore the fourth terms are equal; that is,

$$DF' = DF.$$

$\therefore$ F' coincides with F.

$\therefore$ EF prolonged passes through O.

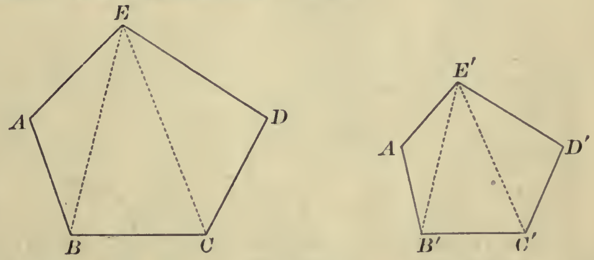
$\therefore$ AB, CD, and EF prolonged meet in the point O.

Q. E. D.

Similar Polygons.

Proposition XII. Theorem.

331. *If two polygons are composed of the same number of triangles, similar each to each, and similarly placed, the polygons are similar.*



In the two polygons $ABCDE$ and $A'B'C'D'E'$, let the triangles AEB, BEC, CED be similar respectively to the triangles $A'E'B'$, $B'E'C'$, $C'E'D'$.

To prove $ABCDE$ similar to $A'B'C'D'E'$.

Proof. $\angle A = \angle A'$, § 319

(*being homologous* ∠s *of similar* △s).

Also, $\angle ABE = \angle A'B'E'$ § 319

and $\angle EBC = \angle E'B'C'$

By adding, $\angle ABC = \angle A'B'C'$.

In like manner we may prove $\angle BCD = \angle B'C'D'$, etc.

Hence the two polygons are mutually equiangular.

Now

$$\frac{AE}{A'E'} = \frac{AB}{A'B'} = \left(\frac{EB}{E'B'}\right) = \frac{BC}{B'C'} = \left(\frac{EC}{E'C'}\right) = \frac{CD}{C'D'} = \frac{ED}{E'D'},$$

(*the homologous sides of similar* △s *are proportional*).

Hence the homologous sides of the polygons are proportional.

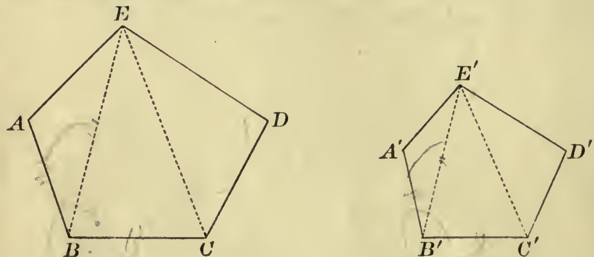
Therefore the polygons are similar, § 319

(*having their homologous* ∠s *equal, and their homologous sides proportional*).

Q. E. D.

Proposition XIII. Theorem.

332. *If two polygons are similar, they are composed of the same number of triangles, similar each to each, and similarly placed.*



Let the polygons $ABCDE$ and $A'B'C'D'E'$ be similar.

From two homologous vertices, as E and E', draw diagonals EB, EC, and $E'B'$, $E'C'$.

To prove $\triangle EAB, EBC, ECD$

similar respectively to $\triangle E'A'B', E'B'C', E'C'D'$.

Proof. In the $\triangle EAB$ and $E'A'B'$,

$$\angle A = \angle A', \qquad \S\ 319$$

(being homologous $\angle$ of similar polygons);

and
$$\frac{AE}{A'E'} = \frac{AB}{A'B'}, \qquad \S\ 319$$

(being homologous sides of similar polygons).

$\therefore \triangle EAB$ and $E'A'B'$ are similar, § 326

(having an $\angle$ of the one equal to an $\angle$ of the other, and the including sides proportional).

Also, $$\angle ABC = \angle A'B'C', \qquad (1)$$

(being homologous $\angle$ of similar polygons).

And $$\angle ABE = \angle A'B'E', \qquad (2)$$

(being homologous $\angle$ of similar $\triangle$).

Subtract (2) from (1),

$$\angle EBC = \angle E'B'C'. \qquad \text{Ax. 3}$$

Now
$$\frac{EB}{E'B'} = \frac{AB}{A'B'},$$
(being homologous sides of similar △).

And
$$\frac{BC}{B'C'} = \frac{AB}{A'B'},$$
(being homologous sides of similar polygons).

$$\therefore \frac{EB}{E'B'} = \frac{BC}{B'C'}.$$ Ax. 1

$\therefore$ △ EBC and $E'B'C'$ are similar, § 326

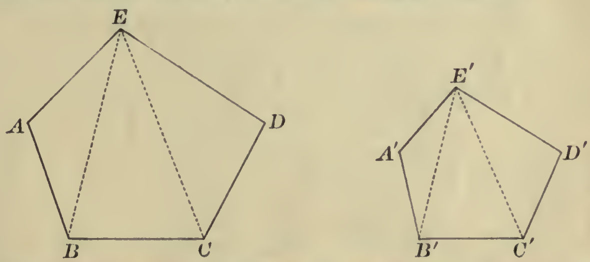
(having an ∠ of the one equal to an ∠ of the other, and the including sides proportional).

In like manner we may prove △ ECD and $E'C'D'$ similar.

Q. E. D.

PROPOSITION XIV. THEOREM.

333. *The perimeters of two similar polygons have the same ratio as any two homologous sides.*



Let the two similar polygons be $ABCDE$ and $A'B'C'D'E'$, and let P and P' represent their perimeters.

To prove $P : P' = AB : A'B'$.

$AB : A'B' = BC : B'C' = CD : C'D'$, etc., § 319

(the homologous sides of similar polygons are proportional).

$\therefore AB + BC$, etc. : $A'B' + B'C'$, etc. $= AB : A'B'$, § 303

(in a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent).

That is, $P : P' = AB : A'B'$.

Q. E. D.

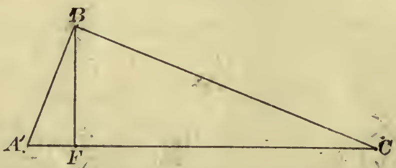
NUMERICAL PROPERTIES OF FIGURES.

PROPOSITION XV. THEOREM.

334. *If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse:*

I. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

II. *Each leg of the right triangle is a mean proportional between the hypotenuse and its adjacent segment.*



In the right triangle ABC, let BF be drawn from the vertex of the right angle B, perpendicular to AC.

I. *To prove* $AF : BF = BF : FC$.

Proof. In the rt. $\triangle$ BAF and BAC

the acute $\angle A$ is common.

Hence the $\triangle$ are similar. § 323

In the rt. $\triangle$ BCF and BCA

the acute $\angle C$ is common.

Hence the $\triangle$ are similar. § 323

Now as the rt. $\triangle$ ABF and CBF are both similar to ABC, they are similar to each other.

In the similar $\triangle$ ABF and CBF,

AF, the shortest side of the one,
: BF, the shortest side of the other,
: : BF, the medium side of the one,
: FC, the medium side of the other.

II. *To prove* $AC : AB = AB : AF$,

and $AC : BC = BC : FC$.

In the similar $\triangle$ ABC and ABF,

AC, the longest side of the one,
: AB, the longest side of the other,
: : AB, the shortest side of the one,
: AF, the shortest side of the other.

Also in the similar $\triangle$ ABC and FBC,

AC, the longest side of the one,
: BC, the longest side of the other,
: : BC, the medium side of the one,
: FC, the medium side of the other. Q. E. D.

335. COR. 1. *The squares of the two legs of a right triangle are proportional to the adjacent segments of the hypotenuse.*

The proportions in II. give, by § 295,

$$\overline{AB}^2 = AC \times AF, \text{ and } \overline{BC}^2 = AC \times CF.$$

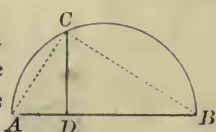
By dividing one by the other, we have

$$\frac{\overline{AB}^2}{\overline{BC}^2} = \frac{AC \times AF}{AC \times CF} = \frac{AF}{CF}.$$

336. COR. 2. *The squares of the hypotenuse and either leg are proportional to the hypotenuse and adjacent segment.*

For
$$\frac{\overline{AC}^2}{\overline{AB}^2} = \frac{AC \times AC}{AC \times AF} = \frac{AC}{AF}.$$

337. COR. 3. An angle inscribed in a semicircle is a right angle (§ 264). Therefore,



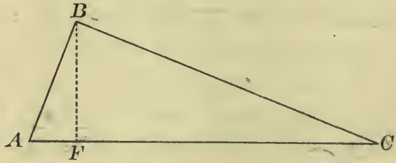
I. *The perpendicular from any point in the circumference to the diameter of a circle is a mean proportional between the segments of the diameter.*

II. *The chord drawn from the point to either extremity of the diameter is a mean proportional between the diameter and the adjacent segment.*

REMARK. The pairs of corresponding sides in similar triangles may be called *longest*, *shortest*, *medium*, to enable the beginner to see quickly these pairs; but he must not forget that two sides are homologous, not because they appear to be the longest or the shortest sides, but because they lie opposite corresponding equal angles.

Proposition XVI. Theorem.

338. *The sum of the squares of the two legs of a right triangle is equal to the square of the hypotenuse.*



Let ABC ***be a right triangle with its right angle at*** B.

To prove $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$.

Proof. Draw $BF \perp$ to AC.

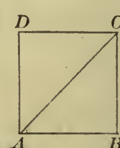
Then $\overline{AB}^2 = AC \times AF$ § 334

and $\overline{BC}^2 = AC \times CF$

By adding, $\overline{AB}^2 + \overline{BC}^2 = AC(AF + CF) = \overline{AC}^2$. Q. E. D.

339. Cor. *The square of either leg of a right triangle is equal to the difference of the squares of the hypotenuse and the other leg.*

340. Scholium. The ratio of the diagonal of a square to the side is the incommensurable number $\sqrt{2}$. For if AC is the diagonal of the square $ABCD$, then

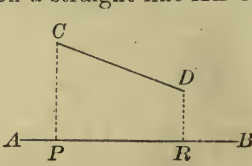


$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2, \text{ or } \overline{AC}^2 = 2\,\overline{AB}^2.$$

Divide by $\overline{AB}^2$, we have $\dfrac{\overline{AC}^2}{\overline{AB}^2} = 2$, or $\dfrac{AC}{AB} = \sqrt{2}$.

Since the square root of 2 is incommensurable, the diagonal and side of a square are two incommensurable lines.

341. *The projection of a line CD* upon a straight line AB is that part of the line AB comprised between the perpendiculars CP and DR let fall from the extremities of CD. Thus, PR is the projection of CD upon AB.



Proposition XVII. Theorem.

342. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other upon that side.*

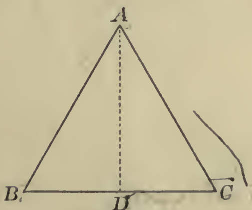


Fig. 1.

Fig. 2.

Let C be an acute angle of the triangle ABC, and DC the projection of AC upon BC.

To prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times DC.$

Proof. If D fall upon the base (Fig. 1),

$$DB = BC - DC;$$

If D fall upon the base produced (Fig. 2),

$$DB = DC - BC.$$

In either case,

$$\overline{DB}^2 = \overline{BC}^2 + \overline{DC}^2 - 2\,BC \times DC.$$

Add $\overline{AD}^2$ to both sides of this equality, and we have

$$\overline{AD}^2 + \overline{DB}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{DC}^2 - 2\,BC \times DC.$$

But $\overline{AD}^2 + \overline{DB}^2 = \overline{AB}^2,$ § 338

and $\overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2,$

(*the sum of the squares of the two legs of a rt. △ is equal to the square of the hypotenuse*).

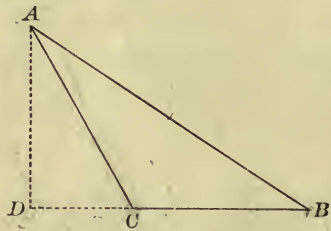
Put $\overline{AB}^2$ and $\overline{AC}^2$ for their equals in the above equality,

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times DC.$$

Q. E. D.

Proposition XVIII. Theorem.

343. *In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides and the projection of the other upon that side.*



Let C be the obtuse angle of the triangle ABC, and CD be the projection of AC upon BC produced.

To prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times DC.$

Proof. $DB = BC + DC.$

Squaring, $\overline{DB}^2 = \overline{BC}^2 + \overline{DC}^2 + 2\,BC \times DC.$

Add $\overline{AD}^2$ to both sides, and we have

$$\overline{AD}^2 + \overline{DB}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{DC}^2 + 2\,BC \times DC.$$

But $\overline{AD}^2 + \overline{DB}^2 = \overline{AB}^2,$ § 338

and $\overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2,$

(*the sum of the squares of the two legs of a rt.* $\triangle$ *is equal to the square of the hypotenuse*).

Put $\overline{AB}^2$ and $\overline{AC}^2$ for their equals in the above equality,

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times DC.$$

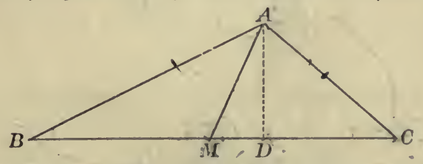
Q. E. D.

NOTE. The last three theorems enable us to compute the lengths of the altitudes if the lengths of the three sides of a triangle are known.

Proposition XIX. Theorem.

344. I. *The sum of the squares of two sides of a triangle is equal to twice the square of half the third side increased by twice the square of the median upon that side.*

II. *The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the median upon that side.*



In the triangle ABC let AM be the median, and MD the projection of AM upon the side BC. Also let AB be greater than AC.

To prove I. $\overline{AB}^2 + \overline{AC}^2 = 2\overline{BM}^2 + 2\overline{AM}^2$.

II. $\overline{AB}^2 - \overline{AC}^2 = 2BC \times MD$.

Proof. Since $AB > AC$, the $\angle AMB$ will be obtuse, and the $\angle AMC$ will be acute. § 153

Then $\overline{AB}^2 = \overline{BM}^2 + \overline{AM}^2 + 2BM \times MD$, § 343

(*in any obtuse △ the square of the side opposite the obtuse ∠ is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides and the projection of the other on that side*);

and $\overline{AC}^2 = \overline{MC}^2 + \overline{AM}^2 - 2MC \times MD$, § 342

(*in any △ the square of the side opposite an acute ∠ is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other upon that side*).

Add these two equalities, and observe that $BM = MC$.

Then $\overline{AB}^2 + \overline{AC}^2 = 2\overline{BM}^2 + 2\overline{AM}^2$.

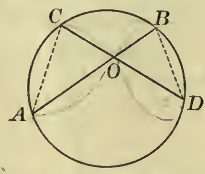
Subtract the second equality from the first.

Then $\overline{AB}^2 - \overline{AC}^2 = 2BC \times MD$. Q. E. D.

Note. This theorem enables us to compute the lengths of the medians if the lengths of the three sides of the triangle are known.

Proposition XX. Theorem.

345. *If any chord is drawn through a fixed point within a circle, the product of its segments is constant in whatever direction the chord is drawn.*



Let any two chords AB and CD intersect at O.

To prove $OA \times OB = OD \times OC$.

Proof. Draw AC and BD.

In the $\triangle$ AOC and BOD,

$\angle C = \angle B$, § 263

(*each being measured by $\frac{1}{2}$ arc AD*).

$\angle A = \angle D$, § 263

(*each being measured by $\frac{1}{2}$ arc BC*).

$\therefore$ the $\triangle$ are similar, § 322

(*two $\triangle$ are similar when two $\angle$ of the one are equal to two $\angle$ of the other*)

Whence OA, the longest side of the one,

: OD, the longest side of the other,

:: OC, the shortest side of the one,

: OB, the shortest side of the other.

$\therefore OA \times OB = OD \times OC$. § 295

Q. E. D

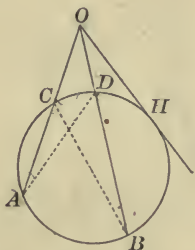
346. Scholium. This proportion may be written

$$\frac{OA}{OD} = \frac{OC}{OB}, \text{ or } \frac{OA}{OD} = \frac{1}{\frac{OB}{OC}};$$

that is, the ratio of two corresponding segments is equal to the *reciprocal* of the ratio of the other two corresponding segments. In this case the segments are said to be *reciprocally proportional.*

Proposition XXI. Theorem.

347. *If from a fixed point without a circle a secant is drawn, the product of the secant and its external segment is constant in whatever direction the secant is drawn.*



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Let OA and OB be two secants drawn from point O.

To prove $OA \times OC = OB \times OD$.

Proof. Draw BC and AD.

In the $\triangle$ OAD and OBC

$\angle O$ is common,

$\angle A = \angle B$, § 263

(each being measured by $\frac{1}{2}$ arc CD).

$\therefore$ the two $\triangle$ are similar, § 322

(two $\triangle$ are similar when two $\angle$ of the one are equal to two $\angle$ of the other).

Whence OA, the longest side of the one,
: OB, the longest side of the other,
: : OD, the shortest side of the one,
: OC, the shortest side of the other.

$\therefore OA \times OC = OB \times OD$. § 295

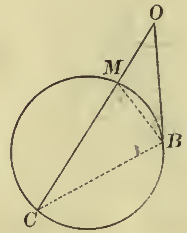
Q. E. D.

Remark. The above proportion continues true if the secant OB turns about O until B and D approach each other indefinitely. Therefore, by the theory of limits, it is true when B and D coincide at H. Whence, $OA \times OC = \overline{OH}^2$.

This truth is demonstrated directly in the next theorem.

Proposition XXII. Theorem.

348. *If from a point without a circle a secant and a tangent are drawn, the tangent is a mean proportional between the whole secant and the external segment.*



Let OB be a tangent and OC a secant drawn from the point O to the circle MBC.

To prove $OC : OB = OB : OM$.

Proof. Draw BM and BC.

In the $\triangle$ OBM and OBC

$\angle O$ is common.

$\angle OBM$ is measured by $\frac{1}{2}$ arc MB, § 269
(being an $\angle$ formed by a tangent and a chord).

$\angle C$ is measured by $\frac{1}{2}$ arc BM, § 263
(being an inscribed $\angle$).

$\therefore \angle OBM = \angle C$.

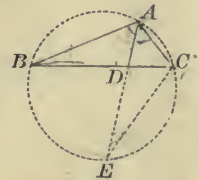
$\therefore$ $\triangle$ OBC and OBM are similar, § 322
(having two $\angle$ of the one equal to two $\angle$ of the other).

Whence OC, the longest side of the one,
: OB, the longest side of the other,
: : OB, the shortest side of the one,
: OM, the shortest side of the other.

Q. E. D.

PROPOSITION XXIII. THEOREM.

349. *The square of the bisector of an angle of a triangle is equal to the product of the sides of this angle diminished by the product of the segments determined by the bisector upon the third side of the triangle.*



Let AD bisect the angle BAC of the triangle ABC.

To prove $\overline{AD}^2 = AB \times AC - DB \times DC.$

Proof. Circumscribe the $\odot$ ABC about the $\triangle$ ABC.

Produce AD to meet the circumference in E, and draw EC.

Then in the $\triangle$ ABD and AEC,

$\angle BAD = \angle CAE$, Hyp.

$\angle B = \angle E$, § 263

(each being measured by ½ the arc AC).

$\therefore$ $\triangle$ ABD and AEC are similar, § 322

(two $\triangle$ are similar if two $\angle$ of the one are equal respectively to two $\angle$ of the other).

Whence AB, the longest side of the one,
: AE, the longest side of the other,
: : AD, the shortest side of the one,
: AC, the shortest side of the other.

$\therefore AB \times AC = AD \times AE$ § 295

$= AD(AD + DE)$

$= \overline{AD}^2 + AD \times DE.$

But $AD \times DE = DB \times DC,$ § 345

(the product of the segments of a chord drawn through a fixed point in a $\odot$ is constant).

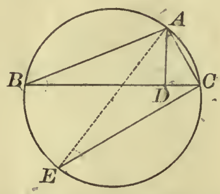
$\therefore AB \times AC = \overline{AD}^2 + DB \times DC.$

Whence $\overline{AD}^2 = AB \times AC - DB \times DC.$ Q. E. D.

NOTE. This theorem enables us to compute the lengths of the bisectors of the angles of a triangle if the lengths of the sides are known.

Proposition XXIV. Theorem.

350. *In any triangle the product of two sides is equal to the product of the diameter of the circumscribed circle by the altitude upon the third side.*



Let ABC be a triangle, AD the altitude, and ABC the circle circumscribed about the triangle ABC.

Draw the diameter AE, and draw EC.

To prove $AB \times AC = AE \times AD$.

Proof. In the ▲ ABD and AEC,

$\angle BDA$ is a rt. $\angle$, Cons.

$\angle ECA$ is a rt. $\angle$, § 264

(*being inscribed in a semicircle*),

and $\angle B = \angle E$. § 263

$\therefore$ ▲ ABD and AEC are similar, § 323

(*two rt.* ▲ *having an acute* $\angle$ *of the one equal to an acute* $\angle$ *of the other are similar*).

Whence AB, the longest side of the one,
: AE, the longest side of the other,
: : AD, the shortest side of the one,
: AC, the shortest side of the other.

$\therefore AB \times AC = AE \times AD$. § 295

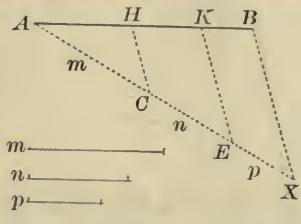
Q. E. D.

Note. This theorem enables us to compute the length of the radius of a circle circumscribed about a triangle, if the lengths of the three sides of the triangle are known.

PROBLEMS OF CONSTRUCTION.

PROPOSITION XXV. PROBLEM.

351. *To divide a given straight line into parts proportional to any number of given lines.*



Let AB, m, n, and p, be given straight lines.

To divide AB into parts proportional to m, n, and p.

Construction. Draw AX, making an acute $\angle$ with AB.

On AX take $AC = m$, $CE = n$, $EX = p$.

Draw BX.

From E and C draw EK and $CH \parallel$ to BX.

K and H are the division points required.

Proof. $$\left(\frac{AK}{AE}\right) = \frac{AH}{AC} = \frac{HK}{CE} = \frac{KB}{EX},$$ § 309

(*a line drawn through two sides of a $\triangle \parallel$ to the third side divides those sides proportionally*).

$$\therefore AH : HK : KB = AC : CE : EX.$$

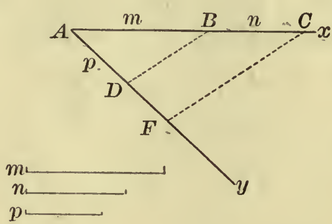
Substitute m, n, and p for their equals AC, CE, and EX.

Then $AH : HK : KB = m : n : p$.

Q.E.F

PROPOSITION XXVI. PROBLEM.

352. *To find a fourth proportional to three given straight lines.*



Let the three given lines be m, n, ***and*** p.

To find a fourth proportional to m, n, *and* p.

Draw Ax and Ay containing any acute angle.

Construction. On Ax take AB equal to m, $BC = n$.

On Ay take $AD = p$.

Draw BD.

From C draw $CF \parallel$ to BD, to meet Ay at F.

DF is the fourth proportional required.

Proof. $AB : BC = AD : DF$, § 309

(*a line drawn through two sides of a* $\triangle \parallel$ *to the third side divides those sides proportionally*).

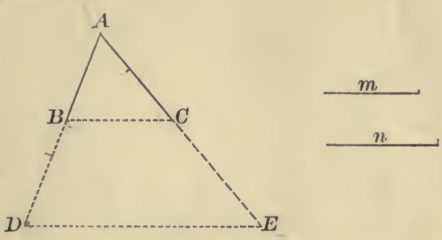
Substitute m, n, and p for their equals AB, BC, and AD.

Then $m : n = p : DF$.

Q. E. F.

Proposition XXVII. Problem.

353. *To find a third proportional to two given straight lines.*



Let m and n be the two given straight lines.

To find a third proportional to m and n.

Construction. Construct any acute angle A,

and take $AB = m$, $AC = n$.

Produce AB to D, making $BD = AC$.

Join BC.

Through D draw $DE \parallel$ to BC to meet AC produced at E.

CE is the third proportional to AB and AC.

Proof. $AB : BD = AC : CE$. § 309

(*a line drawn through two sides of a $\triangle \parallel$ to the third side divides those sides proportionally*).

Substitute, in the above proportion, AC for its equal BD.

Then $AB : AC = AC : CE$.

That is, $m : n = n : CE$.

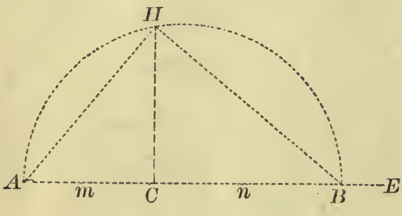
Q. E. F.

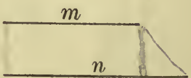
Ex. 217. Construct x; if (1) $x = \frac{ab}{c}$, (2) $x = \frac{a^2}{c}$.

Special Cases: (1) $a = 2$, $b = 3$, $c = 4$; (2) $a = 3$, $b = 7$, $c = 11$; (3) $a = 2$, $c = 3$; (4) $a = 3$, $c = 5$; (5) $a = 2c$.

Proposition XXVIII. Problem.

354. *To find a mean proportional between two given straight lines.*





Let the two given lines be m and n.

To find a mean proportional between m and n.

Construction. On the straight line AE

take $AC = m$, and $CB = n$.

On AB as a diameter describe a semi-circumference.

At C erect the $\perp$ CH to meet the circumference at H.

CH is a mean proportional between m and n.

Proof. $\therefore AC : CH = CH : CB$, § 337

(*the $\perp$ let fall from a point in a circumference to the diameter of a circle is a mean proportional between the segments of the diameter*).

Substitute for AC and CB their equals m and n.

Then $m : CH = CH : n$.

Q. E. F.

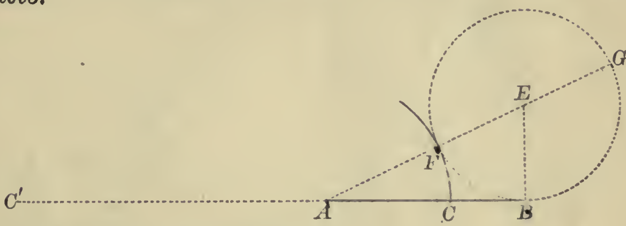
355. A straight line is divided *in extreme and mean ratio*, when one of the segments is a mean proportional between the whole line and the other segment.

Ex. 218. Construct x if $x = \sqrt{ab}$.

Special Cases: (1) $a = 2$, $b = 3$; (2) $a = 1$, $b = 5$; (3) $a = 3$, $b = 7$.

PROPOSITION XXIX. PROBLEM.

356. *To divide a given line in extreme and mean ratio.*



Let AB be the given line.

To divide AB in extreme and mean ratio.

Construction. At B erect a $\perp$ BE equal to one-half of AB.

From E as a centre, with a radius equal to EB, describe a $\odot$.

Draw AE, meeting the circumference in F and G.

On AB take $AC = AF$.

On BA produced take $AC' = AG$.

Then AB is divided internally at C and externally at C' in extreme and mean ratio.

Proof. $AG : AB = AB : AF$, § 348

(*if from a point without a* $\odot$ *a secant and a tangent are drawn, the tangent is a mean proportional between the whole secant and the external segment*).

Then by § 301 and § 300,

$$AG - AB : AB = AB - AF : AF, \qquad (1)$$

$$AG + AB : AG = AB + AF : AB. \qquad (2)$$

By construction $FG = 2\,EB = AB$.

$\therefore\ AG - AB = AG - FG = AF = AC.$

Hence (1) becomes

$AC : AB = BC : AC;$

or, by inversion, $AB : AC = AC : BC.$

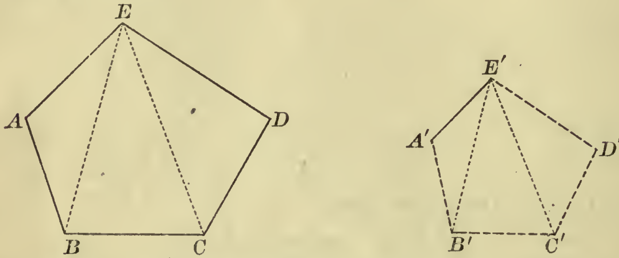
Again, since $C'A = AG = AB + AF,$

(2) becomes $C'B : C'A = C'A : AB.$

Q. E. F.

Proposition XXX. Problem.

357. *Upon a given line homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.*



Let $A'E'$ be the given line homologous to AE of the given polygon $ABCDE$.

To construct on $A'E'$ a polygon similar to the given polygon.

Construction. From E draw the diagonals EB and EC.

From E' draw $E'B'$, $E'C'$, and $E'D'$,

making $\measuredangle$ $A'E'B'$, $B'E'C'$, and $C'E'D'$ equal respectively to $\measuredangle$ AEB, BEC, and CED.

From A' draw $A'B'$, making $\angle E'A'B' = \angle EAB$, and meeting $E'B'$ at B'.

From B' draw $B'C'$, making $\angle E'B'C' = \angle EBC$, and meeting $E'C'$ at C'.

From C' draw $C'D'$, making $\angle E'C'D' = \angle ECD$, and meeting $E'D'$ at D'.

Then $A'B'C'D'E'$ is the required polygon.

Proof. The corresponding $\triangle$ ABE and $A'B'E'$, EBC and $E'B'C'$, ECD and $E'C'D'$ are similar, § 322

(*two $\triangle$ are similar if they have two $\measuredangle$ of the one equal respectively to two $\measuredangle$ of the other*).

Then the two polygons are similar, § 331

(*two polygons composed of the same number of $\triangle$ similar to each other and similarly placed, are similar*).

Q. E. F.

PROBLEMS OF COMPUTATION.

219. To compute the altitudes of a triangle in terms of its sides.

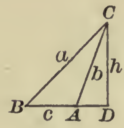


FIG. 1.

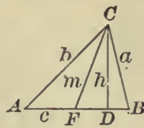


FIG. 2.

At least one of the angles A or B is acute. Suppose it is the angle B.

In the $\triangle\, CDB$, $\quad h^2 = a^2 - \overline{BD}^2.$ § 338

In the $\triangle\, ABC$, $\quad b^2 = a^2 + c^2 - 2c \times BD.$ § 342

Whence, $\quad BD = \dfrac{a^2 + c^2 - b^2}{2c}.$

Hence
$$h^2 = a^2 - \frac{(a^2 + c^2 - b^2)^2}{4c^2} = \frac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{4c^2}$$
$$= \frac{(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}{4c^2}$$
$$= \frac{\{(a+c)^2 - b^2\}\{b^2 - (a-c)^2\}}{4c^2}$$
$$= \frac{(a+b+c)(a+c-b)(b+a-c)(b-a+c)}{4c^2}.$$

Let $\quad a + b + c = 2s.$

Then
$$a + c - b = 2(s - b),$$
$$b + a - c = 2(s - c),$$
$$b - a + c = 2(s - a).$$

Hence
$$h^2 = \frac{2s \times 2(s-a) \times 2(s-b) \times 2(s-c)}{4c^2}.$$

By simplifying, and extracting the square root,
$$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

220. To compute the medians of a triangle in terms of its sides.

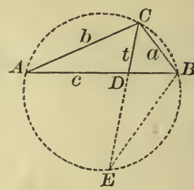
By § 344, $\quad a^2 + b^2 = 2m^2 + 2\left(\dfrac{c}{2}\right)^2.$ (Fig. 2)

Whence $\quad 4m^2 = 2(a^2 + b^2) - c^2.$

$$\therefore m = \tfrac{1}{2}\sqrt{2(a^2 + b^2) - c^2}.$$

$\overline{s(s-a)(s-b)(s-c)} = S$ area of a Δ

221. To compute the bisectors of a triangle in terms of the sides.



By § 349, $t^2 = ab - AD \times BD.$

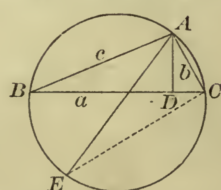
By § 313, $\dfrac{AD}{b} = \dfrac{BD}{a} = \dfrac{AD + BD}{a + b} = \dfrac{c}{a + b}.$

$$\therefore AD = \frac{bc}{a + b}, \text{ and } BD = \frac{ac}{a + b}.$$

Whence
$$t^2 = ab - \frac{abc^2}{(a + b)^2}$$
$$= ab\left(1 - \frac{c^2}{(a + b)^2}\right)$$
$$= \frac{ab\{(a + b)^2 - c^2\}}{(a + b)^2}$$
$$= \frac{ab(a + b + c)(a + b - c)}{(a + b)^2}$$
$$= \frac{ab \times 2s \times 2(s - c)}{(a + b)^2}.$$

Whence
$$t = \frac{2}{a + b}\sqrt{abs(s - c)}.$$

222. To compute the radius of the circle circumscribed about a triangle in terms of the sides of the triangle.



By § 350, $AB \times AC = AE \times AD,$

or
$$bc = 2R \times AD.$$

But
$$AD = \frac{2}{a}\sqrt{s(s - a)(s - b)(s - c)}.$$

Whence
$$R = \frac{abc}{4\sqrt{s(s - a)(s - b)(s - c)}}.$$

223. If the sides of a triangle are 3, 4, and 5, is the angle opposite 5 right, acute, or obtuse?

224. If the sides of a triangle are 7, 9, and 12, is the angle opposite 12 right, acute, or obtuse? ob.

225. If the sides of a triangle are 7, 9, and 11, is the angle opposite 11 right, acute, or obtuse? ac

226. The legs of a right triangle are 8 inches and 12 inches; find the lengths of the projections of these legs upon the hypotenuse, and the distance of the vertex of the right angle from the hypotenuse.

227. If the sides of a triangle are 6 inches, 9 inches, and 12 inches, find the lengths (1) of the altitudes; (2) of the medians; (3) of the bisectors; (4) of the radius of the circumscribed circle.

THEOREMS.

228. Any two altitudes of a triangle are inversely proportional to the corresponding bases.

229. Two circles touch at P. Through P three lines are drawn, meeting one circle in A, B, C, and the other in A', B', C', respectively. Prove that the triangles ABC, $A'B'C'$ are similar.

230. Two chords AB, CD intersect at M, and A is the middle point of the arc CD. Prove that the product $AB \times AM$ remains the same if the chord AB is made to turn about the fixed point A.

HINT. Draw the diameter AE, join BE, and compare the triangles thus formed.

231. The sum of the squares of the segments of two perpendicular chords is equal to the square of the diameter of the circle.

If AB, CD are the chords, draw the diameter BE, join AC, ED, BD, and prove that $AC = ED$. Apply § 338.

232. In a parallelogram $ABCD$, a line DE is drawn, meeting the diagonal AC in F, the side BC in G, and the side AB produced in E. Prove that $\overline{DF}^2 = FG \times FE$.

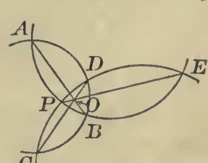
233. The tangents to two intersecting circles drawn from any point in their common chord produced, are equal. (§ 348.)

234. The common chord of two intersecting circles, if produced, will bisect their common tangents. (§ 348.)

235. If two circles touch each other, their common tangent is a mean proportional between their diameters.

HINT. Let AB be the common tangent. Draw the diameters AC, BD. Join the point of contact P to A, B, C, and D. Show that APD and BPC are straight lines $\perp$ to each other, and compare $\triangle$ ABC, ABD.

236. If three circles intersect one another, the common chords all pass through the same point.

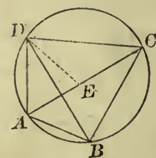


HINT. Let two of the chords AB and CD meet at O. Join the point of intersection E to O, and suppose that EO produced meets the same two circles at two different points P and Q. Then prove that $OP = OQ$; hence, that the points P and Q coincide.

237. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the circumference of the smaller circle.

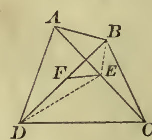
Hint. Draw any two of the chords, join the points where they meet the circumferences, and prove that the $\triangle$s thus formed are similar.

238. In an inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides.



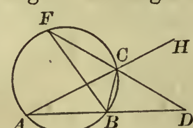
Hint. Draw DE, making $\angle CDE = \angle ADB$. The $\triangle$s ABD and CDE are similar. Also the $\triangle$s BCD and ADE are similar.

239. The sum of the squares of the four sides of any quadrilateral is equal to the sum of the squares of the diagonals, increased by four times the square of the line joining the middle points of the diagonals.



Hint. Join the middle points F, E, of the diagonals. Draw EB and ED. Apply § 344 to the $\triangle$s ABC and ADC, add the results, and eliminate $\overline{BE}^2 + \overline{DE}^2$ by applying § 343 to the $\triangle\, BDE$.

240. The square of the bisector of an exterior angle of a triangle is equal to the product of the external segments determined by the bisector upon one of the sides, diminished by the product of the other two sides.



Hint. Let CD bisect the exterior $\angle BCH$ of the $\triangle ABC$. Circumscribe a $\odot$ about the $\triangle$, produce DC to meet the circumference in F, and draw BF. Prove $\triangle$s ACD, BCF similar. Apply § 347.

241. If a point O is joined to the vertices of a triangle ABC, and through any point A' in OA a line parallel to AB is drawn, meeting OB at B', and then through B' a line parallel to BC, meeting OC at C', and C' is joined to A', the triangle $A'B'C'$ will be similar to the triangle ABC.

242. If the line of centres of two circles meets the circumferences at the points A, B, C, D, and meets the common exterior tangent at P, then $PA \times PD = PB \times PC$.

243. The line of centres of two circles meets the common exterior tangent at P, and a secant is drawn from P, cutting the circles at the consecutive points E, F, G, H. Prove that $PE \times PH = PF \times PG$.

6 : 4 :: x : 90
4x = 540
x = 135

NUMERICAL EXERCISES.

244. A line is drawn parallel to a side AB of a triangle ABC, and cutting AC in D, BC in E. If $AD:DC = 2:3$, and $AB = 20$ inches, find DE. 15

245. The sides of a triangle are 9, 12, 15. Find the segments made by bisecting the angles. (§ 313.)

246. A tree casts a shadow 90 feet long, when a vertical rod 6 feet high casts a shadow 4 feet long. How high is the tree?

247. The bases of a trapezoid are represented by a, b, and the altitude by h. Find the altitudes of the two triangles formed by producing the legs till they meet.

248. The sides of a triangle are 6, 7, 8. In a similar triangle the side homologous to 8 is equal to 40. Find the other two sides. 30

249. The perimeters of two similar polygons are 200 feet and 300 feet. If a side of the first polygon is 24 feet, find the homologous side of the second polygon. 36

250. How long must a ladder be to reach a window 24 feet high, if the lower end of the ladder is 10 feet from the side of the house?

251. If the side of an equilateral triangle $= a$, find the altitude.

252. If the altitude of an equilateral triangle $= h$, find the side.

253. Find the lengths of the longest and the shortest chord that can be drawn through a point 6 inches from the centre of a circle whose radius is equal to 10 inches. 16

254. The distance from the centre of a circle to a chord 10 inches long is 12 inches. Find the distance from the centre to a chord 24 inches long. 5

255. The radius of a circle is 5 inches. Through a point 3 inches from the centre a diameter is drawn, and also a chord perpendicular to the diameter. Find the length of this chord, and the distance from one end of the chord to the ends of the diameter. 9

256. The radius of a circle is 6 inches. Through a point 10 inches from the centre tangents are drawn. Find the lengths of the tangents, and also of the chord joining the points of contact.

257. If a chord 8 inches long is 3 inches distant from the centre of the circle, find the radius and the distances from the end of the chord to the ends of the diameter which bisects the chord. 5 9

258. The radius of a circle is 13 inches. Through a point 5 inches from the centre any chord is drawn. What is the product of the two segments of the chord? 125 What is the length of the shortest chord that can be drawn through the point? 24

259. From the end of a tangent 20 inches long a secant is drawn through the centre of the circle. If the exterior segment of this secant is 8 inches, find the radius of the circle. 21

260. The radius of a circle is 9 inches; the length of a tangent is 12 inches. Find the length of a secant drawn from the extremity of the tangent to the centre of the circle. 10.8 +

261. The radii of two circles are 8 inches and 3 inches, and the distance between their centres is 15 inches. Find the lengths of their common tangents. 14 +

262. Find the segments of a line 10 inches long divided in extreme and mean ratio.

263. The sides of a triangle are 4, 5, 6. Is the largest angle acute, right, or obtuse? a

Problems.

264. To divide one side of a given triangle into segments proportional to the adjacent sides. (§ 313.)

265. To produce a line AB to a point C so that $AB : AC = 3 : 5$.

266. To find in one side of a given triangle a point whose distances from the other sides shall be to each other in a given ratio.

267. Given an obtuse triangle; to draw a line from the vertex of the obtuse angle to the opposite side which shall be a mean proportional between the segments of that side.

268. Through a given point P within a given circle to draw a chord AB so that $AP : BP = 2 : 3$.

269. To draw through a given point P in the arc subtended by a chord AB a chord which shall be bisected by AB.

270. To draw through a point P, exterior to a given circle, a secant PAB so that $PA : AB = 4 : 3$.

271. To draw through a point P, exterior to a given circle, a secant PAB so that $\overline{AB}^2 = PA \times PB$.

272. To find a point P in the arc subtended by a given chord AB so that $PA : PB = 3 : 1$.

273. To draw through one of the points of intersection of two circles a secant so that the two chords that are formed shall be to each other in the ratio of $3:5$.

274. To divide a line into three parts proportional to $2, \frac{3}{4}, \frac{1}{2}$.

275. Having given the greater segment of a line divided in extreme and mean ratio, to construct the line.

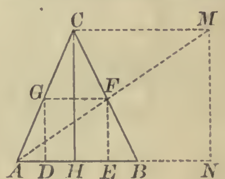
276. To construct a circle which shall pass through two given points and touch a given straight line.

277. To construct a circle which shall pass through a given point and touch two given straight lines.

278. To inscribe a square in a semicircle.

279. To inscribe a square in a given triangle.

Hint. Suppose the problem solved, and $DEFG$ the inscribed square. Draw $CM \parallel$ to AB, and let AF produced meet CM in M. Draw CH and $MN \perp$ to AB, and produce AB to meet MN at N. The $\triangle ACM$, AGF are similar; also the $\triangle AMN$, AFE are similar. By these triangles show that the figure $CMNH$ is a square. By constructing this square, the point F can be found.



280. To inscribe in a given triangle a rectangle similar to a given rectangle.

281. To inscribe in a circle a triangle similar to a given triangle.

282. To inscribe in a given semicircle a rectangle similar to a given rectangle.

283. To circumscribe about a circle a triangle similar to a given triangle.

284. To construct the expression, $x = \frac{2\,abc}{de}$; that is, $\frac{2\,ab}{d} \times \frac{c}{e}$.

285. To construct two straight lines, having given their sum and their ratio.

286. To construct two straight lines, having given their difference and their ratio.

287. Having given two circles, with centres O and O', and a point A in their plane, to draw through the point A a straight line, meeting the circumferences at B and C, so that $AB : AC = 1 : 2$.

Hint. Suppose the problem solved, join OA and produce it to D, making $OA : AD = 1 : 2$. Join DC; $\triangle OAB$, ADC are similar.

BOOK IV.

AREAS OF POLYGONS.

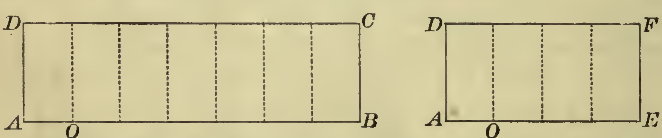
358. The *area* of a surface is the *numerical measure* of the surface referred to the *unit of surface.*

The unit of surface is a square whose side is a *unit of length;* as the *square inch*, the *square foot*, etc.

359. *Equivalent figures* are figures having equal areas.

Proposition I. Theorem.

360. ***The areas of two rectangles having equal altitudes are to each other as their bases.***



Let the two rectangles be AC and AF, having the same altitude AD.

To prove $$\frac{\text{rect. } AC}{\text{rect. } AF} = \frac{AB}{AE}.$$

Proof. Case I. *When AB and AE are commensurable.*

Suppose AB and AE have a common measure, as AO, which is contained in AB seven times and in AE four times.

Then $$\frac{AB}{AE} = \frac{7}{4}. \qquad (1)$$

Apply this measure to AB and AE, and at the several points of division erect $\perp$s.

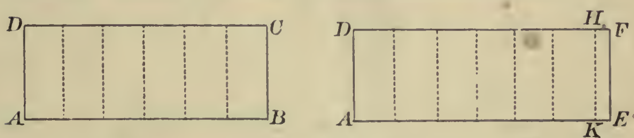
The rect. AC will be divided into seven rectangles, and the rect. AF will be divided into four rectangles.

These rectangles are all equal. § 186

Hence $$\frac{\text{rect. } AC}{\text{rect. } AF} = \frac{7}{4}. \quad (2)$$

From (1) and (2) $$\frac{\text{rect. } AC}{\text{rect. } AF} = \frac{AB}{AE}. \quad \text{Ax. 1}$$

CASE II. *When AB and AE are incommensurable.*



Divide AB into any number of equal parts, and apply one of them to AE as often as it will be contained in AE.

Since AB and AE are incommensurable, a certain number of these parts will extend from A to a point K, leaving a remainder KE less than one of the parts.

Draw $KH \parallel$ to EF.

Since AB and AK are commensurable,

$$\frac{\text{rect. } AH}{\text{rect. } AC} = \frac{AK}{AB}. \quad \text{Case I.}$$

These ratios continue equal, as the unit of measure is indefinitely diminished, and approach indefinitely the limiting ratios $\frac{\text{rect. } AF}{\text{rect. } AC}$ and $\frac{AE}{AB}$ respectively.

$$\therefore \frac{\text{rect. } AF}{\text{rect. } AC} = \frac{AE}{AB}, \quad \text{§ 260}$$

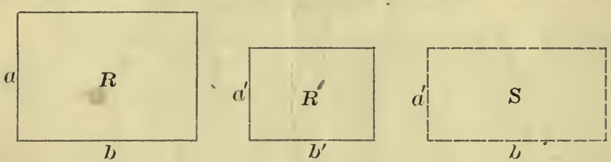
(*if two variables are constantly equal, and each approaches a limit, the limits are equal*). Q. E. D.

361. COR. *The areas of two rectangles having equal bases are to each other as their altitudes.* For AB and AE may be considered as the altitudes, AD and AD as the bases.

NOTE. In propositions relating to *areas*, the words "rectangle," "triangle," etc., are often used for "area of rectangle," "area of triangle," etc.

Proposition II. Theorem.

362. *The areas of two rectangles are to each other as the products of their bases by their altitudes.*



Let R and R' be two rectangles, having for their bases b and b', and for their altitudes a and a'.

To prove $$\frac{R}{R'} = \frac{a \times b}{a' \times b'}.$$

Proof. Construct the rectangle S, with its base the same as that of R, and its altitude the same as that of R'.

Then $$\frac{R}{S} = \frac{a}{a'},$$ § 361

(*rectangles having equal bases are to each other as their altitudes*);

and $$\frac{S}{R'} = \frac{b}{b'},$$ § 360

(*rectangles having equal altitudes are to each other as their bases*).

By multiplying these two equalities,

$$\frac{R}{R'} = \frac{a \times b}{a' \times b'}.$$ Q. E. D.

Ex. 288. Find the ratio of a rectangular lawn 72 yards by 49 yards to a grass turf 18 inches by 14 inches.

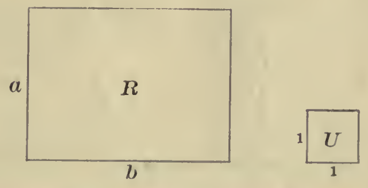
Ex. 289. Find the ratio of a rectangular courtyard $18\frac{1}{2}$ yards by $15\frac{1}{2}$ yards to a flagstone 31 inches by 18 inches.

Ex. 290. A square and a rectangle have the same perimeter, 100 yards. The length of the rectangle is 4 times its breadth. Compare their areas.

Ex. 291. On a certain map the linear scale is 1 inch to 5 miles. How many acres are represented on this map by a square the perimeter of which is 1 inch ?

PROPOSITION III. THEOREM.

363. *The area of a rectangle is equal to the product of its base and altitude.*



Let** R **be the rectangle,** b **the base, and** a **the altitude; and let** U **be a square whose side is equal to the linear unit.

To prove the area of $R = a \times b$.

$$\frac{R}{U} = \frac{a \times b}{1 \times 1} = a \times b, \qquad \S\ 362$$

(*two rectangles are to each other as the product of their bases and altitudes*).

But $$\frac{R}{U} = \text{the area of } R. \qquad \S\ 358$$

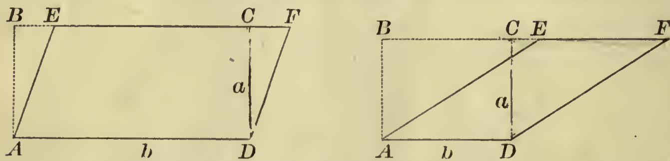
$$\therefore \text{ the area of } R = a \times b. \qquad \text{Q. E. D.}$$

364. SCHOLIUM. When the base and altitude each contain the linear unit an integral number of times, this proposition is rendered evident by dividing the figure into squares, each

equal to the unit of measure. Thus, if the base contain seven linear units, and the altitude four, the figure may be divided into twenty-eight squares, each equal to the unit of measure; and the area of the figure equals 7×4 units of surface.

PROPOSITION IV. THEOREM.

365. *The area of a parallelogram is equal to the product of its base and altitude.*



Let $AEFD$ be a parallelogram, AD its base, and CD its altitude.

To prove the area of the ▱ $AEFD = AD \times CD$.

Proof. From A draw $AB \parallel$ to DC to meet FE produced.

Then the figure $ABCD$ will be a rectangle, with the same base and altitude as the ▱ $AEFD$.

In the rt. $\triangle$s ABE and DCF

$AB = CD$ and $AE = DF$, § 179
(*being opposite sides of a* ▱).

$\therefore \triangle ABE = \triangle DCF$, § 161
(*two rt.* $\triangle$s *are equal when the hypotenuse and a side of the one are equal respectively to the hypotenuse and a side of the other*).

Take away the $\triangle DCF$, and we have left the rect. $ABCD$.

Take away the $\triangle ABE$, and we have left the ▱ $AEFD$.

$\therefore$ rect. $ABCD \approx$ ▱ $AEFD$. Ax. 3

But the area of the rect. $ABCD = a \times b$, § 363

$\therefore$ the area of the ▱ $AEFD = a \times b$. Ax. 1

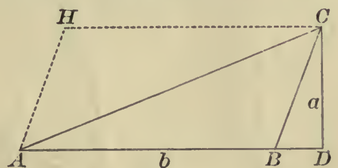
Q. E. D.

366. COR. 1. *Parallelograms having equal bases and equal altitudes are equivalent.*

367. COR. 2. *Parallelograms having equal bases are to each other as their altitudes; parallelograms having equal altitudes are to each other as their bases; any two parallelograms are to each other as the products of their bases by their altitudes.*

PROPOSITION V. THEOREM.

368. *The area of a triangle is equal to one-half of the product of its base by its altitude.*



Let ABC be a triangle, AB its base, and DC its altitude.

To prove the area of the $\triangle\ ABC = \frac{1}{2}\ AB \times DC$.

Proof. From C draw $CH \parallel$ to AB.

From A draw $AH \parallel$ to BC.

The figure $ABCH$ is a parallelogram, § 168

(*having its opposite sides parallel*),

and AC is its diagonal.

$\therefore \triangle\ ABC = \triangle\ AHC$, § 178

(*the diagonal of a* ▱ *divides it into two equal* $\triangle$s).

The area of the ▱ $ABCH$ is equal to the product of its base by its altitude. § 365

Therefore the area of one-half the ▱, that is, the area of the $\triangle\ ABC$, is equal to one-half the product of its base by its altitude.

Hence, the area of the $\triangle\ ABC = \frac{1}{2} AB \times DC$.

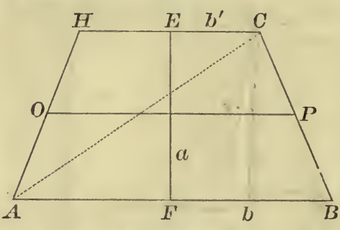
Q. E. D.

369. COR. 1. *Triangles having equal bases and equal altitudes are equivalent.*

370. COR. 2. *Triangles having equal bases are to each other as their altitudes; triangles having equal altitudes are to each other as their bases; any two triangles are to each other as the product of their bases by their altitudes.*

Proposition VI. Theorem.

371. *The area of a trapezoid is equal to one-half the sum of the parallel sides multiplied by the altitude.*



Let ABCH be a trapezoid, and EF the altitude.

To prove area of $ABCH = \frac{1}{2}(HC + AB)\,EF$.

Proof. Draw the diagonal AC.

Then the area of the $\triangle\, ABC = \frac{1}{2}(AB \times EF)$, § 368

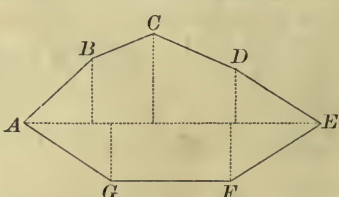
and the area of the $\triangle\, AHC = \frac{1}{2}(HC \times EF)$.

By adding, area of $ABCH = \frac{1}{2}(AB + HC)\,EF$. Q. E. D.

372. Cor. *The area of a trapezoid is equal to the product of the median by the altitude.* For, by § 191, OP is equal to $\frac{1}{2}(HC + AB)$; and hence

the area of $ABCH = OP \times EF$.

373. Scholium. The area of an irregular polygon may be found by dividing the polygon into triangles, and by finding the area of each of these triangles separately. But the method generally employed in practice is to draw the longest diagonal,

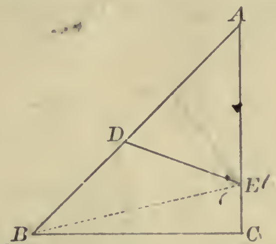


and to let fall perpendiculars upon this diagonal from the other angular points of the polygon.

The polygon is thus divided into right triangles and trapezoids; the sum of the areas of these figures will be the area of the polygon.

PROPOSITION VII. THEOREM.

374. *The areas of two triangles which have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Let the triangles ABC **and** ADE **have the common angle** A.

To prove $$\frac{\triangle ABC}{\triangle ADE} = \frac{AB \times AC}{AD \times AE}.$$

Proof. Draw BE.

Now $$\frac{\triangle ABC}{\triangle ABE} = \frac{AC}{AE},$$

and $$\frac{\triangle ABE}{\triangle ADE} = \frac{AB}{AD}, \qquad \text{§ 370}$$

(*△ having the same altitude are to each other as their bases*).

By multiplying these equalities,

$$\frac{\triangle ABC}{\triangle ADE} = \frac{AB \times AC}{AD \times AE}.$$

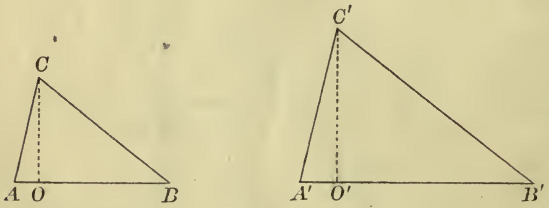
Q. E. D.

Ex. 292. The areas of two triangles which have an angle of the one supplementary to an angle of the other are to each other as the products of the sides including the supplementary angles.

Comparison of Polygons.

Proposition VIII. Theorem.

375. *The areas of two similar triangles are to each other as the squares of any two homologous sides.*



Let the two triangles be ACB **and** $A'C'B'$.

To prove $$\frac{\triangle ACB}{\triangle A'C'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Draw the perpendiculars CO and $C'O'$.

Then $$\frac{\triangle ACB}{\triangle A'C'B'} = \frac{AB \times CO}{A'B' \times C'O'} = \frac{AB}{A'B'} \times \frac{CO}{C'O'}, \qquad \S\ 370$$

(*two* $\triangle$ *are to each other as the products of their bases by their altitudes*).

But $$\frac{AB}{A'B'} = \frac{CO}{C'O'}, \qquad \S\ 328$$

(*the homologous altitudes of similar* $\triangle$ *have the same ratio as their homologous bases*).

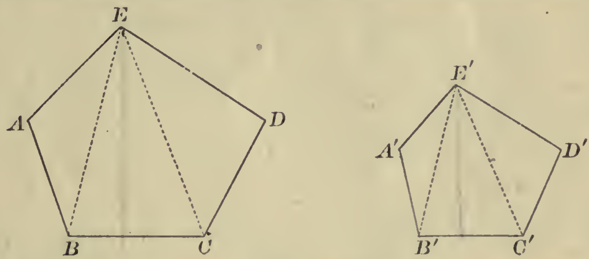
Substitute, in the above equality, for $\frac{CO}{C'O'}$ its equal $\frac{AB}{A'B'}$;

then $$\frac{\triangle ACB}{\triangle A'C'B'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Q. E. D

Proposition IX. Theorem.

376. *The areas of two similar polygons are to each other as the squares of any two homologous sides.*



Let S and S' denote the areas of the two similar polygons ABC, etc., and $A'B'C'$, etc.

To prove $S : S' = \overline{AB}^2 : \overline{A'B'}^2$.

Proof. By drawing all the diagonals from the homologous vertices E and E', the two similar polygons are divided into triangles similar and similarly placed. § 332

$$\therefore \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\triangle ABE}{\triangle A'B'E'} = \left(\frac{\overline{BE}^2}{\overline{B'E'}^2}\right) = \frac{\triangle BCE}{\triangle B'C'E'}$$

$$= \left(\frac{\overline{CE}^2}{\overline{C'E'}^2}\right) = \frac{\triangle CDE}{\triangle C'D'E'}, \qquad \text{§ 375}$$

(*similar* $\triangle$s *are to each other as the squares of any two homologous sides*).

That is, $$\frac{\triangle ABE}{\triangle A'B'E'} = \frac{\triangle BCE}{\triangle B'C'E'} = \frac{\triangle CDE}{\triangle C'D'E'}.$$

$$\therefore \frac{\triangle s\ ABE + BCE + CDE}{\triangle s\ A'B'E' + B'C'E' + C'D'E'} = \frac{\triangle ABE}{\triangle A'B'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}, \qquad \text{§ 303}$$

(*in a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent*).

$$\therefore S : S' = \overline{AB}^2 : \overline{A'B'}^2.$$

Q. E. D.

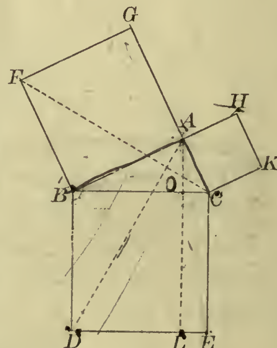
377. Cor. 1. *The areas of two similar polygons are to each other as the squares of any two homologous lines.*

378. Cor. 2. *The homologous sides of two similar polygons have the same ratio as the square roots of their areas.*

PROPOSITION X. THEOREM.

379. *The square described on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.*

Let BE, CH, AF, be squares on the three sides of the right triangle ABC.



To prove $\overline{BC}^2 \approx \overline{AB}^2 + \overline{AC}^2$.

Proof. Through A draw $AL \parallel$ to CE, and draw AD and FC.

Since $\angle s$ BAC, BAG, and CAH are rt. $\angle s$, CAG and BAH are straight lines.

Since $BD = BC$, being sides of the same square, and $BA = BF$, for the same reason, and since $\angle ABD = \angle FBC$, each being the sum of a rt. $\angle$ and the $\angle ABC$,

the $\triangle ABD = \triangle FBC$. § 150

Now the rectangle BL is double the $\triangle ABD$,

(*having the same base BD, and the same altitude, the distance between the $\parallel$s AL and BD*),

and the square AF is double the $\triangle FBC$,

(*having the same base FB, and the same altitude, the distance between the $\parallel$s FB and GC*).

Hence the rectangle BL is equivalent to the square AF.

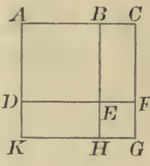
In like manner, by joining AE and BK, it may be proved that the rectangle CL is equivalent to the square CH.

Therefore the square BE, which is the sum of the rectangles BL and CL, is equivalent to the sum of the squares CH and AF. Q. E. D.

380. COR. *The square on either leg of a right triangle is equivalent to the difference of the squares on the hypotenuse and the other leg.*

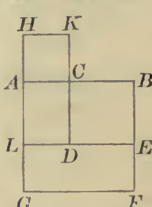
Ex. 293. The square constructed upon the sum of two straight lines is equivalent to the sum of the squares constructed upon these two lines, increased by twice the rectangle of these lines.

Let AB and BC be the two straight lines, and AC their sum. Construct the squares $ACGK$ and $ABED$ upon AC and AB respectively. Prolong BE and DE until they meet KG and CG respectively. Then we have the square $EFGH$, with sides each equal to BC. Hence, the square $ACGK$ is the sum of the squares $ABED$ and $EFGH$, and the rectangles $DEHK$ and $BCFE$, the dimensions of which are equal to AB and BC.



Ex. 294. The square constructed upon the difference of two straight lines is equivalent to the sum of the squares constructed upon these two lines, diminished by twice the rectangle of these lines.

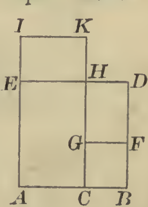
Let AB and AC be the two straight lines, and BC their difference. Construct the square $ABFG$ upon AB, the square $ACKH$ upon AC, and the square $BEDC$ upon BC (as shown in the figure). Prolong ED until it meets AG in L.



The dimensions of the rectangles $LEFG$ and $HKDL$ are AB and AC, and the square $BCDE$ is evidently the difference between the whole figure and the sum of these rectangles; that is, the square constructed upon BC is equivalent to the sum of the squares constructed upon AB and AC diminished by twice the rectangle of AB and AC.

Ex. 295. The difference between the squares constructed upon two straight lines is equivalent to the rectangle of the sum and difference of these lines.

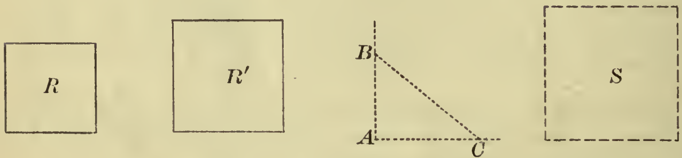
Let $ABDE$ and $BCGF$ be the squares constructed upon the two straight lines AB and BC. The difference between these squares is the polygon $ACGFDE$, which polygon, by prolonging CG to H, is seen to be composed of the rectangles $ACHE$ and $GFDH$. Prolong AE and CH to I and K respectively, making EI and HK each equal to BC, and draw IK. The rectangles $GFDH$ and $EHKI$ are equal. The difference between the squares $ABDE$ and $BCGF$ is then equivalent to the rectangle $ACKI$, which has for dimensions $AI = AB + BC$, and $EH = AB - BC$.



Problems of Construction.

Proposition XI. Problem.

381. *To construct a square equivalent to the sum of two given squares.*



Let R and R' be two given squares.

To construct a square equivalent to $R' + R$.

Construction. Construct the rt. $\angle A$.

Take AC equal to a side of R',

AB equal to a side of R; and draw BC.

Construct the square S, having each of its sides equal to BC.

S is the square required.

Proof. $\overline{BC}^2 \approx \overline{AC}^2 + \overline{AB}^2$, § 379

(*the square on the hypotenuse of a rt.* $\triangle$ *is equivalent to the sum of the squares on the two sides*).

$$\therefore S \approx R' + R.$$

Q. E. F.

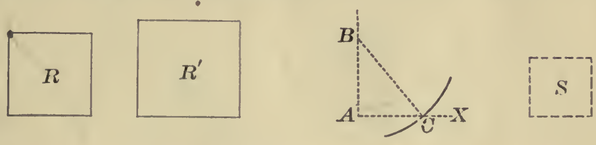
Ex. 296. If the perimeter of a rectangle is 72 feet, and the length is equal to twice the width, find the area.

Ex. 297. How many tiles 9 inches long and 4 inches wide will be required to pave a path 8 feet wide surrounding a rectangular court 120 feet long and 36 feet wide?

Ex. 298. The bases of a trapezoid are 16 feet and 10 feet; each leg is equal to 5 feet. Find the area of the trapezoid.

Proposition XII. Problem.

382. *To construct a square equivalent to the difference of two given squares.*



Let R be the smaller square and R′ the larger.

To construct a square equivalent to $R' - R$.

Construction. Construct the rt. $\angle A$.

Take AB equal to a side of R.

From B as a centre, with a radius equal to a side of R', describe an arc cutting the line AX at C.

Construct the square S, having each of its sides equal to AC.

S is the square required.

Proof. $\overline{AC}^2 \approx \overline{BC}^2 - \overline{AB}^2$; § 380

(*the square on either leg of a rt. △ is equivalent to the difference of the squares on the hypotenuse and the other leg*).

$$\therefore S \approx R' - R.$$

Q. E. F.

Ex. 299. Construct a square equivalent to the sum of two squares whose sides are 3 inches and 4 inches.

Ex. 300. Construct a square equivalent to the difference of two squares whose sides are $2\frac{1}{2}$ inches and 2 inches.

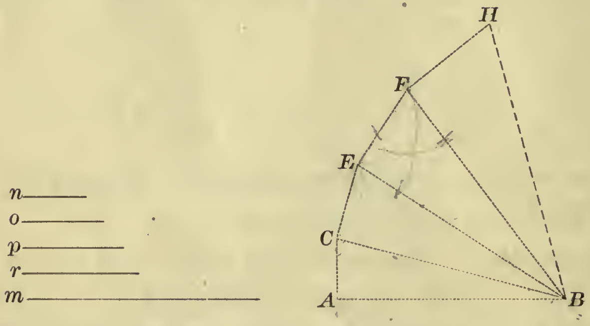
Ex. 301. Find the side of a square equivalent to the sum of two squares whose sides are 24 feet and 32 feet.

Ex. 302. Find the side of a square equivalent to the difference of two squares whose sides are 24 feet and 40 feet.

Ex. 303. A rhombus contains 100 square feet, and the length of one diagonal is 10 feet. Find the length of the other diagonal.

PROPOSITION XIII. PROBLEM.

383. *To construct a square equivalent to the sum of any number of given squares.*



Let m, n, o, p, r be sides of the given squares.

To construct a square $\Leftrightarrow m^2 + n^2 + o^2 + p^2 + r^2$.

Construction. Take $AB = m$.

Draw $AC = n$ and $\perp$ to AB at A, and draw BC.

Draw $CE = o$ and $\perp$ to BC at C, and draw BE.

Draw $EF = p$ and $\perp$ to BE at E, and draw BF.

Draw $FH = r$ and $\perp$ to BF at F, and draw BH.

The square constructed on BH is the square required.

Proof. $\overline{BH}^2 \Leftrightarrow \overline{FH}^2 + \overline{BF}^2,$

$\Leftrightarrow \overline{FH}^2 + \overline{EF}^2 + \overline{EB}^2,$

$\Leftrightarrow \overline{FH}^2 + \overline{EF}^2 + \overline{EC}^2 + \overline{CB}^2,$

$\Leftrightarrow \overline{FH}^2 + \overline{EC}^2 + \overline{EF}^2 + \overline{CA}^2 + \overline{AB}^2,$ § 379

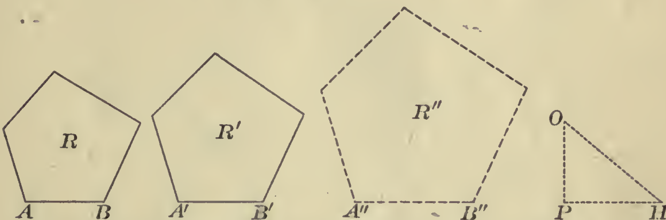
(*the sum of the squares on the two legs of a rt. $\triangle$ is equivalent to the square on the hypotenuse*).

That is, $\overline{BH}^2 \Leftrightarrow m^2 + n^2 + o^2 + p^2 + r^2.$

Q. E. F.

PROPOSITION XIV. PROBLEM.

384. *To construct a polygon similar to two given similar polygons and equivalent to their sum.*



Let R and R′ be two similar polygons, and AB and A′B′ two homologous sides.

To construct a similar polygon equivalent to $R + R'$.

Construction. Construct the rt. $\angle P$.

Take $PH = A'B'$, and $PO = AB$.

Draw OH, and take $A''B'' = OH$.

Upon $A''B''$, homologous to AB, construct R'' similar to R.

Then R'' is the polygon required.

Proof. $\overline{PO}^2 + \overline{PH}^2 = \overline{OH}^2$, $\therefore \overline{AB}^2 + \overline{A'B'}^2 = \overline{A''B''}^2$.

Now $$\frac{R}{R''} = \frac{\overline{AB}^2}{\overline{A''B''}^2},$$

and $$\frac{R'}{R''} = \frac{\overline{A'B'}^2}{\overline{A''B''}^2}, \qquad \S\ 376$$

(*similar polygons are to each other as the squares of their homologous sides*).

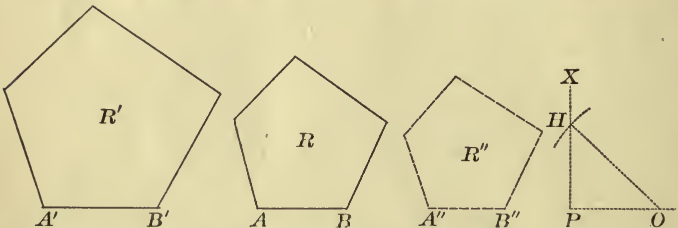
By addition, $$\frac{R + R'}{R''} = \frac{\overline{AB}^2 + \overline{A'B'}^2}{\overline{A''B''}^2} = 1.$$

$$\therefore R'' \approx R + R'.$$

Q. E. F

PROPOSITION XV. PROBLEM.

385. *To construct a polygon similar to two given similar polygons and equivalent to their difference.*



Let R and R' be two similar polygons, and AB and $A'B'$ two homologous sides.

To construct a similar polygon equivalent to $R' - R$.

Construction. Construct the rt. $\angle P$,

and take $PO = AB$.

From O as a centre, with a radius equal to $A'B'$, describe an arc cutting PX at H, and join OH.

Take $A''B'' = PH$, and on $A''B''$, homologous to AB, construct R'' similar to R.

Then R'' is the polygon required.

Proof. $\overline{PH}^2 = \overline{OH}^2 - \overline{OP}^2$, $\therefore \overline{A''B''}^2 = \overline{A'B'}^2 - \overline{AB}^2$.

Now $$\frac{R'}{R''} = \frac{\overline{A'B'}^2}{\overline{A''B''}^2},$$

and $$\frac{R}{R''} = \frac{\overline{AB}^2}{\overline{A''B''}^2}, \qquad \S\ 376$$

(*similar polygons are to each other as the squares of their homologous sides*).

By subtraction,

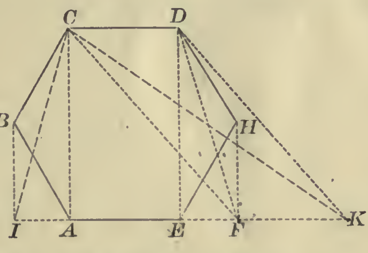
$$\frac{R' - R}{R''} = \frac{\overline{A'B'}^2 - \overline{AB}^2}{\overline{A''B''}^2} = 1.$$

$$\therefore R'' \approxeq R' - R.$$

Q. E. F

PROPOSITION XVI. PROBLEM.

386. *To construct a triangle equivalent to a given polygon.*



Let ABCDHE be the given polygon.

To construct a triangle equivalent to the given polygon.

Construction. From D draw DE,

and from H draw $HF \parallel$ to DE.

Produce AE to meet HF at F, and draw DF.

Again, draw CF, and draw $DK \parallel$ to CF to meet AF produced at K, and draw CK.

In like manner continue to reduce the number of sides of the polygon until we obtain the $\triangle CIK$.

Proof. The polygon $ABCDF$ has one side less than the polygon $ABCDHE$, but the two are equivalent.

For the part $ABCDE$ is common,

and the $\triangle DEF \approx \triangle DEH$, § 369

(*for the base DE is common, and their vertices F and H are in the line FH $\parallel$ to the base*).

The polygon $ABCK$ has one side less than the polygon $ABCDF$, but the two are equivalent.

For the part $ABCF$ is common,

and the $\triangle CFK \approx \triangle CFD$, § 369

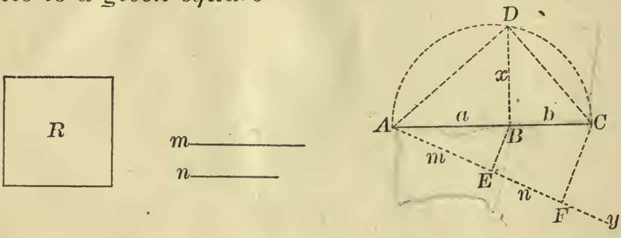
(*for the base CF is common, and their vertices K and D are in the line KD $\parallel$ to the base*).

In like manner the $\triangle CIK \approx ABCK$.

Q. E. F.

Proposition XVII. Problem.

387. *To construct a square which shall have a given ratio to a given square.*



Let R be the given square, and $\frac{n}{m}$ the given ratio.

To construct a square which shall be to R as n is to m.

Construction. Take AB equal to a side of R, and draw Ay, making any acute angle with AB.

On Ay take $AE = m$, $EF = n$, and join EB.

Draw $FC \parallel$ to EB to meet AB produced at C.

On AC as a diameter describe a semicircle.

At B erect the $\perp$ BD, meeting the semicircumference at D.

Then BD is a side of the square required.

Proof. Denote AB by a, BC by b, and BD by x.

Now $a : x = x : b$; that is, $x^2 = ab$. § 337

Hence, a^2 will have the same ratio to x^2 and to ab.

Therefore $a^2 : x^2 = a^2 : ab = a : b$.

But $a : b = m : n$, § 309

(*a straight line drawn through two sides of a $\triangle$, parallel to the third side, divides those sides proportionally*).

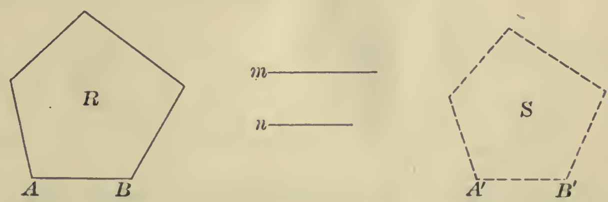
Therefore $a^2 : x^2 = m : n$.

By inversion, $x^2 : a^2 = n : m$.

Hence the square on BD will have the same ratio to R as n has to m. Q. E. F.

PROPOSITION XVIII. PROBLEM.

388. *To construct a polygon similar to a given polygon and having a given ratio to it.*



Let R ***be the given polygon and*** $\frac{n}{m}$ ***the given ratio.***

To construct a polygon similar to R, which shall be to R as n is to m.

Construction. Find a line $A'B'$, such that the square constructed upon it shall be to the square constructed upon AB as n is to m. § 387

Upon $A'B'$ as a side homologous to AB, construct the polygon S similar to R.

Then S is the polygon required.

Proof. $S : R = \overline{A'B'}^2 : \overline{AB}^2$, § 376

(*similar polygons are to each other as the squares of their homologous sides*).

But $\overline{A'B'}^2 : \overline{AB}^2 = n : m$. Cons.

Therefore $S : R = n : m$.

Q. E. F.

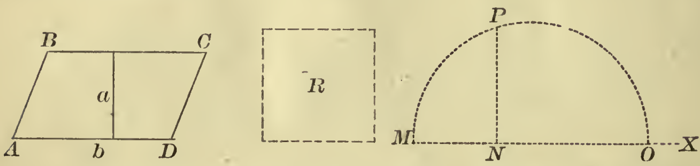
Ex. 304. Find the area of a right triangle if the length of the hypotenuse is 17 feet, and the length of one leg is 8 feet.

Ex. 305. Compare the altitudes of two equivalent triangles, if the base of one is three times that of the other.

Ex. 306. The bases of a trapezoid are 8 feet and 10 feet, and the altitude is 6 feet. Find the base of an equivalent rectangle having an equal altitude.

PROPOSITION XIX. PROBLEM.

389. *To construct a square equivalent to a given parallelogram.*



Let $ABCD$ be a parallelogram, b its base, and a its altitude.

To construct a square equivalent to the $\square$ $ABCD$.

Construction. Upon the line MX take $MN = a$, and $NO = b$.

Upon MO as a diameter, describe a semicircle.

At N erect $NP \perp$ to MO, to meet the circumference at P.

Then the square R, constructed upon a line equal to NP, is equivalent to the $\square$ $ABCD$.

Proof. $MN : NP = NP : NO$, § 337

(*a $\perp$ let fall from any point of a circumference to the diameter is a mean proportional between the segments of the diameter*).

$$\therefore \overline{NP}^2 = MN \times NO = a \times b.$$

That is, $R \Leftrightarrow \square ABCD$.

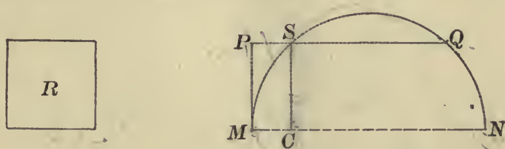
Q. E. F.

390. COR. 1. *A square may be constructed equivalent to a given triangle, by taking for its side a mean proportional between the base and one-half the altitude of the triangle.*

391. COR. 2. *A square may be constructed equivalent to a given polygon, by first reducing the polygon to an equivalent triangle, and then constructing a square equivalent to the triangle.*

PROPOSITION XX. PROBLEM.

392. *To construct a parallelogram equivalent to a given square, and having the sum of its base and altitude equal to a given line.*



Let R be the given square, and let the sum of the base and altitude of the required parallelogram be equal to the given line MN.

To construct a ▱ equivalent to R, with the sum of its base and altitude equal to MN.

Construction. Upon MN as a diameter, describe a semicircle.

At M erect a $\perp$ MP, equal to a side of the given square R.

Draw $PQ \parallel$ to MN, cutting the circumference at S.

Draw $SC \perp$ to MN.

Any ▱ having CM for its altitude and CN for its base is equivalent to R.

Proof. $SC = PM$. §§ 100, 130

$$\therefore \overline{SC}^2 = \overline{PM}^2 = R.$$

But $MC : SC = SC : CN$, § 337

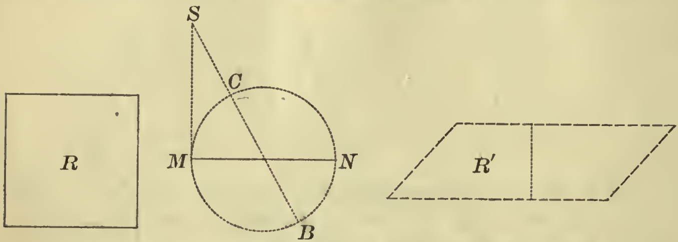
(*a $\perp$ let fall from any point in the circumference to the diameter is a mean proportional between the segments of the diameter*).

Then $\overline{SC}^2 \approx MC \times CN$. Q. E. F.

NOTE. This problem may be stated: *To construct two straight lines the sum and product of which are known.*

Proposition XXI. Problem.

393. *To construct a parallelogram equivalent to a given square, and having the difference of its base and altitude equal to a given line.*



Let R be the given square, and let the difference of the base and altitude of the required parallelogram be equal to the given line MN.

To construct a ▱ equivalent to R, with the difference of the base and altitude equal to MN.

Construction. Upon the given line MN as a diameter, describe a circle.

From M draw MS, tangent to the ⊙, and equal to a side of the given square R.

Through the centre of the ⊙ draw SB intersecting the circumference at C and B.

Then any ▱, as R', having SB for its base and SC for its altitude, is equivalent to R.

Proof. $SB : SM = SM : SC,$ § 348

(*if from a point without a ⊙ a secant and a tangent are drawn, the tangent is a mean proportional between the whole secant and the part without the ⊙*).

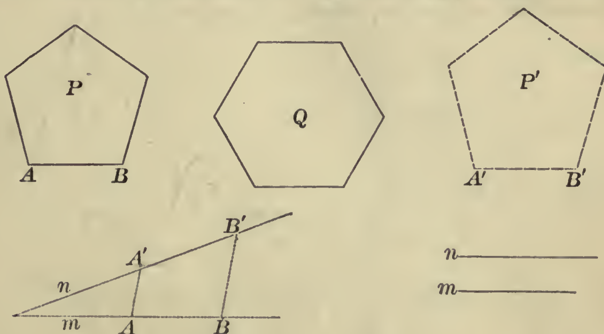
Then $\overline{SM}^2 \approx SB \times SC,$

and the difference between SB and SC is the diameter of the ⊙, that is, MN. Q. E. F.

NOTE. This problem may be stated: *To construct two straight lines the difference and product of which are known.*

Proposition XXII. Problem.

394. *To construct a polygon similar to a given polygon P, and equivalent to a given polygon Q.*



Let P and Q be two polygons, and AB a side of P.

To construct a polygon similar to P and equivalent to Q.

Construction. Find squares equivalent to P and Q, § 391

and let m and n respectively denote their sides.

Find $A'B'$, a fourth proportional to m, n, and AB. § 351

Upon $A'B'$, homologous to AB, construct P' similar to P.

Then P' is the polygon required.

Proof. $m : n = AB : A'B'$, Cons.

$$\therefore m^2 : n^2 = \overline{AB}^2 : \overline{A'B'}^2.$$

But $P \approx m^2$, and $Q \approx n^2$. Cons.

$$\therefore P : Q = m^2 : n^2 = \overline{AB}^2 : \overline{A'B'}^2.$$

But $P : P' = \overline{AB}^2 : \overline{A'B'}^2$, § 376

(*similar polygons are to each other as the squares of their homologous sides*).

$\therefore P : Q = P . P'$. Ax. 1

$\therefore P'$ is equivalent to Q, and is similar to P by construction.

Q. E. F.

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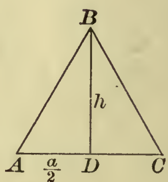
Problems of Computation.

Ex. 307. To find the area of an equilateral triangle in terms of its side.

Denote the side by a, the altitude by h, and the area by S.



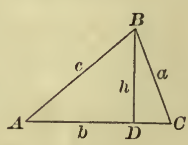
Then $$h^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}.$$

$$\therefore h = \frac{a}{2}\sqrt{3}.$$

But $$S = \frac{a \times h}{2}.$$

$$\therefore S = \frac{a}{2} \times \frac{a\sqrt{3}}{2} = \frac{a^2\sqrt{3}}{4}.$$

Ex. 308. To find the area of a triangle in terms of its sides.



By Ex. 219, $$h = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$$

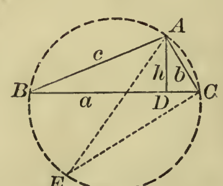
Hence, $$S = \frac{b}{2} \times \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

Ex. 309. To find the area of a triangle in terms of the radius of the circumscribing circle.

If R denote the radius of the circumscribing circle, and h the altitude of the triangle, we have, by Ex. 222,

$$b \times c = 2R \times h.$$



Multiply by a, and we have

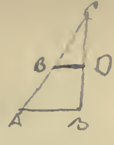
$$a \times b \times c = 2R \times a \times h.$$

But $$a \times h = 2S.$$

$$\therefore a \times b \times c = 4R \times S$$

$$\therefore S = \frac{abc}{4R}.$$

Note. The radius of the circumscribing circle is equal to $\frac{abc}{4S}$.



Theorems.

310. In a right triangle the product of the legs is equal to the product of the hypotenuse and the perpendicular drawn to the hypotenuse from the vertex of the right angle.

311. If ABC is a right triangle, C the vertex of the right angle, BD a line cutting AC in D, then $\overline{BD}^2 + \overline{AC}^2 = \overline{AB}^2 + \overline{DC}^2$.

312. Upon the sides of a right triangle as homologous sides three similar polygons are constructed. Prove that the polygon upon the hypotenuse is equivalent to the sum of the polygons upon the legs.

313. Two isosceles triangles are equivalent if their legs are equal each to each, and the altitude of one is equal to half the base of the other.

314. The area of a circumscribed polygon is equal to half the product of its perimeter by the radius of the inscribed circle.

315. Two parallelograms are equal if two adjacent sides of the one are equal respectively to two adjacent sides of the other, and the included angles are supplementary.

316. Every straight line drawn through the centre of a parallelogram divides it into two equal parts.

317. If the middle points of two adjacent sides of a parallelogram are joined, a triangle is formed which is equivalent to one-eighth of the entire parallelogram.

318. If any point within a parallelogram is joined to the four vertices, the sum of either pair of triangles having parallel bases is equivalent to one-half the parallelogram.

319. The line which joins the middle points of the bases of a trapezoid divides the trapezoid into two equivalent parts.

320. The area of a trapezoid is equal to the product of one of the legs and the distance from this leg to the middle point of the other leg.

321. The lines joining the middle point of the diagonal of a quadrilateral to the opposite vertices divide the quadrilateral into two equivalent parts.

322. The figure whose vertices are the middle points of the sides of any quadrilateral is equivalent to one-half of the quadrilateral.

323. ABC is a triangle, M the middle point of AB, P any point in AB between A and M. If MD is drawn parallel to PC, and meeting BC at D, the triangle BPD is equivalent to one-half the triangle ABC.

NUMERICAL EXERCISES.

324. Find the area of a rhombus, if the sum of its diagonals is 12 feet, and their ratio is 3 : 5.

325. Find the area of an isosceles right triangle if the hypotenuse is 20 feet.

326. In a right triangle, the hypotenuse is 13 feet, one leg is 5 feet. Find the area.

327. Find the area of an isosceles triangle if the base $= b$, and leg $= c$.

328. Find the area of an equilateral triangle if one side $= 8$.

329. Find the area of an equilateral triangle if the altitude $= h$.

330. A house is 40 feet long, 30 feet wide, 25 feet high to the roof, and 35 feet high to the ridge-pole. Find the number of square feet in its entire exterior surface.

331. The sides of a right triangle are as 3 : 4 : 5. The altitude upon the hypotenuse is 12 feet. Find the area.

332. Find the area of a right triangle if one leg $= a$, and the altitude upon the hypotenuse $= h$.

333. Find the area of a triangle if the lengths of the sides are 104 feet, 111 feet, and 175 feet.

334. The area of a trapezoid is 700 square feet. The bases are 30 feet and 40 feet respectively. Find the distance between the bases.

335. $ABCD$ is a trapezium; $AB = 87$ feet, $BC = 119$ feet, $CD = 41$ feet, $DA = 169$ feet, $AC = 200$ feet. Find the area.

336. What is the area of a quadrilateral circumscribed about a circle whose radius is 25 feet, if the perimeter of the quadrilateral is 400 feet? What is the area of a hexagon having an equal perimeter and circumscribed about the same circle?

337. The base of a triangle is 15 feet, and its altitude is 8 feet. Find the perimeter of an equivalent rhombus if the altitude is 6 feet.

338. Upon the diagonal of a rectangle 24 feet by 10 feet a triangle equivalent to the rectangle is constructed. What is its altitude?

339. Find the side of a square equivalent to a trapezoid whose bases are 56 feet and 44 feet, and each leg is 10 feet.

340. Through a point P in the side AB of a triangle ABC, a line is drawn parallel to BC, and so as to divide the triangle into two equivalent parts. Find the value of AP in terms of AB.

341. What part of a parallelogram is the triangle cut off by a line drawn from one vertex to the middle point of one of the opposite sides?

342. In two similar polygons, two homologous sides are 15 feet and 25 feet. The area of the first polygon is 450 square feet. Find the area of the other polygon.

343. The base of a triangle is 32 feet, its altitude 20 feet. What is the area of the triangle cut off by drawing a line parallel to the base and at a distance of 15 feet from the base?

344. The sides of two equilateral triangles are 3 feet and 4 feet. Find the side of an equilateral triangle equivalent to their sum.

345. If the side of one equilateral triangle is equal to the altitude of another, what is the ratio of their areas?

346. The sides of a triangle are 10 feet, 17 feet, and 21 feet. Find the areas of the parts into which the triangle is divided by bisecting the angle formed by the first two sides.

347. In a trapezoid, one base is 10 feet, the altitude is 4 feet, the area is 32 square feet. Find the length of a line drawn between the legs parallel to the base and distant 1 foot from it.

348. If the altitude h of a triangle is increased by a length m, how much must be taken from the base a in order that the area may remain the same?

349. Find the area of a right triangle, having given the segments p, q, into which the hypotenuse is divided by a perpendicular drawn to the hypotenuse from the vertex of the right angle.

PROBLEMS.

350. To construct a triangle equivalent to a given triangle, and having one side equal to a given length l.

351. To transform a triangle into an equivalent right triangle.

352. To transform a triangle into an equivalent isosceles triangle.

353. To transform a triangle ABC into an equivalent triangle, having one side equal to a given length l, and one angle equal to angle BAC.

HINTS. Upon AB (produced if necessary), take $AD = l$, draw $BE \parallel$ to CD, and meeting AC (produced if necessary) at E; $\triangle BED \approx \triangle BEC$.

354. To transform a given triangle into an equivalent right triangle, having one leg equal to a given length.

355. To transform a given triangle into an equivalent right triangle, having the hypotenuse equal to a given length.

356. To transform a given triangle into an equivalent isosceles triangle, having the base equal to a given length.

To construct a triangle equivalent to:

357. The sum of two given triangles.

358. The difference of two given triangles.

359. To transform a given triangle into an equivalent equilateral triangle.

To transform a parallelogram into:

360. A parallelogram having one side equal to a given length.

361. A parallelogram having one angle equal to a given angle.

362. A rectangle having a given altitude.

To transform a square into:

363. An equilateral triangle.

364. A right triangle having one leg equal to a given length.

365. A rectangle having one side equal to a given length.

To construct a square equivalent to:

366. Five-eighths of a given square.

367. Three-fifths of a given pentagon.

368. To draw a line through the vertex of a given triangle so as to divide the triangle into two triangles which shall be to each other as 2 : 3.

369. To divide a given triangle into two equivalent parts by drawing a line through a given point P in one of the sides.

370. To find a point within a triangle, such that the lines joining this point to the vertices shall divide the triangle into three equivalent parts.

371. To divide a given triangle into two equivalent parts by drawing a line parallel to one of the sides.

372. To divide a given triangle into two equivalent parts by drawing a line perpendicular to one of the sides.

373. To divide a given parallelogram into two equivalent parts by drawing a line through a given point in one of the sides.

374. To divide a given trapezoid into two equivalent parts by drawing a line parallel to the bases.

375. To divide a given trapezoid into two equivalent parts by drawing a line through a given point in one of the bases.

Review

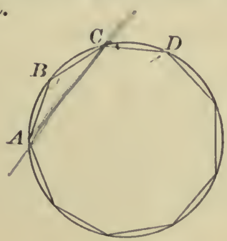
BOOK V.

REGULAR POLYGONS AND CIRCLES.

395. A *regular polygon* is a polygon which is equilateral and equiangular; as, for example, the equilateral triangle, and the square.

Proposition I. Theorem.

396. *An equilateral polygon inscribed in a circle is a regular polygon.*



Let ABC, etc., be an equilateral polygon inscribed in a circle.

To prove the polygon ABC, etc., regular.

Proof. The arcs AB, BC, CD, etc., are equal, § 230
(*in the same ⊙, equal chords subtend equal arcs*).

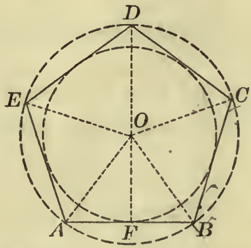
Hence arcs ABC, BCD, etc., are equal, Ax. 6

and the ∠s A, B, C, etc., are equal,
(*being inscribed in equal segments*).

Therefore the polygon ABC, etc., is a regular polygon, being equilateral and equiangular. Q. E. D.

Proposition II. Theorem.

397. *A circle may be circumscribed about, and a circle may be inscribed in, any regular polygon.*



Let $ABCDE$ be a regular polygon.

I. *To prove that a circle may be circumscribed about $ABCDE$.*

Proof. Let O be the centre of the circle passing through A, B, C.

Join OA, OB, OC, and OD.

Since the polygon is equiangular, and the $\triangle OBC$ is isosceles,

$$\angle ABC = \angle BCD$$

and $$\angle OBC = \angle OCB$$

By subtraction, $$\angle OBA = \angle OCD$$

Hence in the $\triangle$s OBA and OCD

the $\angle OBA = \angle OCD$,

the radius $OB =$ the radius OC,

and $AB = CD$. § 395

$\therefore \triangle OAB = \triangle OCD$, § 150

(*having two sides and the included $\angle$ of the one equal to two sides and the included $\angle$ of the other*).

$$\therefore OA = OD.$$

Therefore the circle passing through A, B, and C, also passes through D.

In like manner it may be proved that the circle passing through B, C, and D, also passes through E; and so on through all the vertices in succession.

Therefore a circle described from O as a centre, and with a radius OA, will be circumscribed about the polygon.

II. *To prove that a circle may be inscribed in $ABCDE$.*

Proof. Since the sides of the regular polygon are equal chords of the circumscribed circle, they are equally distant from the centre. § 236

Therefore a circle described from O as a centre, and with the distance from O to a side of the polygon as a radius, will be inscribed in the polygon. Q. E. D.

398. The radius of the circumscribed circle, OA, is called the *radius of the polygon.*

399. The radius of the inscribed circle, OF, is called the *apothem of the polygon.*

400. The common centre O of the circumscribed and inscribed circles is called the *centre of the polygon.*

401. The angle between radii drawn to the extremities of any side, as angle AOB, is called the *angle at the centre of the polygon.*

By joining the centre to the vertices of a regular polygon, the polygon can be decomposed into as many equal isosceles triangles as it has sides. Therefore,

402. Cor. 1. *The angle at the centre of a regular polygon is equal to four right angles divided by the number of sides of the polygon.*

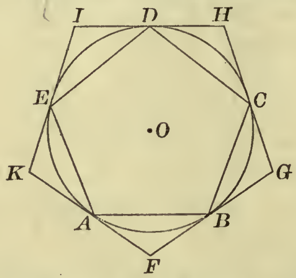
403. Cor. 2. *The radius drawn to any vertex of a regular polygon bisects the angle at the vertex.*

404. Cor. 3. *The interior angle of a regular polygon is the supplement of the angle at the centre.*

For the $\angle ABC = 2\angle ABO = \angle ABO + \angle BAO$. Hence the $\angle ABC$ is the supplement of the $\angle AOB$.

Proposition III. Theorem.

405. *If the circumference of a circle is divided into any number of equal parts, the chords joining the successive points of division form a regular inscribed polygon, and the tangents drawn at the points of division form a regular circumscribed polygon.*



Let the circumference be divided into equal arcs, AB, BC, CD, etc., be chords, FBG, GCH, etc., be tangents.

I. *To prove that $ABCDE$ is a regular polygon.*

Proof. The sides AB, BC, CD, etc., are equal, § 230
(*in the same ⊙ equal arcs are subtended by equal chords*).

Therefore the polygon is regular, § 396
(*an equilateral polygon inscribed in a ⊙ is regular*).

II. *To prove that the polygon $FGHIK$ is a regular polygon.*

Proof. In the ⊿ AFB, BGC, CHD, etc.

$AB = BC = CD$, etc. § 395

Also, $\angle BAF = \angle ABF = \angle CBG = \angle BCG$, etc., § 269
(*being measured by halves of equal arcs*).

Therefore the triangles are all equal isosceles triangles.

Hence $\angle F = \angle G = \angle H$, etc.

Also, $FB = BG = GC = CH$, etc.

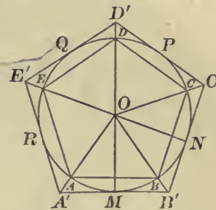
Therefore $FG = GH$, etc.

$\therefore FGHIK$ is a regular polygon. § 395

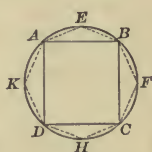
Q. E. D.

406. Cor. 1. *Tangents to a circumference at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.*

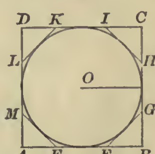
407. Cor. 2. *If a regular polygon is inscribed in a circle, the tangents drawn at the middle points of the arcs subtended by the sides of the polygon form a circumscribed regular polygon, whose sides are parallel to the sides of the inscribed polygon and whose vertices lie on the radii (prolonged) of the inscribed polygon.* For any two corresponding sides, as AB and $A'B'$, perpendicular to OM, are parallel, and the tangents MB' and NB', intersecting at a point equidistant from OM and ON (§ 246), intersect upon the bisector of the $\angle MON$ (§ 163); that is, upon the radius OB.



408. Cor. 3. *If the vertices of a regular inscribed polygon are joined to the middle points of the arcs subtended by the sides of the polygon, the joining lines form a regular inscribed polygon of double the number of sides.*



409. Cor. 4. *If tangents are drawn at the middle points of the arcs between adjacent points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon of double the number of sides is formed.*

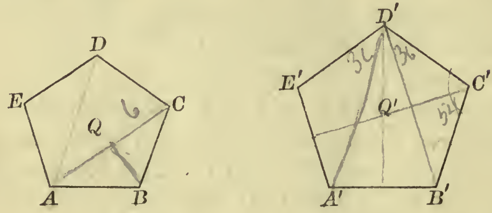


410. Scholium. The perimeter of an inscribed polygon is less than the perimeter of the inscribed polygon of double the number of sides; for each pair of sides of the second polygon is greater than the side of the first polygon which they replace (§ 137).

The perimeter of a circumscribed polygon is greater than the perimeter of the circumscribed polygon of double the number of sides; for every alternate side FG, HI, etc., of the polygon $FGHI$, etc., replaces portions of two sides of the circumscribed polygon $ABCD$, and forms with them a triangle, and one side of a triangle is less than the sum of the other two sides.

Proposition IV. Theorem.

411. *Two regular polygons of the same number of sides are similar.*



Let Q and Q′ be two regular polygons, each having n sides.

To prove *Q and Q′ similar polygons.*

Proof. The sum of the interior ∠s of each polygon is equal to

$$(n-2)\ 2 \text{ rt. } \angle s, \qquad \S\ 205$$

(*the sum of the interior ∠s of a polygon is equal to 2 rt. ∠s taken as many times less 2 as the polygon has sides*).

$$\text{Each angle of either polygon} = \frac{(n-2)\ 2 \text{ rt. } \angle s}{n}, \qquad \S\ 206$$

(*for the ∠s of a regular polygon are all equal, and hence each ∠ is equal to the sum of the ∠s divided by their number*).

Hence the two polygons Q and Q' are mutually equiangular.

Since $AB = BC$, etc., and $A'B' = B'C'$, etc., § 395

$$AB : A'B' = BC : B'C', \text{ etc.}$$

Hence the two polygons have their homologous sides proportional.

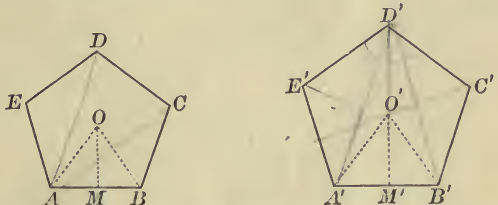
Therefore the two polygons are similar. § 319

Q. E. D.

412. Cor. *The areas of two regular polygons of the same number of sides are to each other as the squares of any two homologous sides.* § 376

PROPOSITION V. THEOREM.

413. *The perimeters of two regular polygons of the same number of sides are to each other as the radii of their circumscribed circles, and also as the radii of their inscribed circles.*



Let P and P' denote the perimeters, O and O' the centres, of the two regular polygons.

From O, O' draw OA, $O'A'$, OB, $O'B'$, and ⊥s OM, $O'M'$.

To prove $P : P' = OA : O'A' = OM : O'M'$.

Proof. Since the polygons are similar, § 411

$$P : P' = AB : A'B'. \qquad \text{§ 333}$$

In the isosceles △s OAB and $O'A'B'$

$$\text{the } \angle O = \text{the } \angle O', \qquad \text{§ 402}$$

$$\text{and } OA : OB = O'A' : O'B'.$$

$$\therefore \text{the △s } OAB \text{ and } O'A'B' \text{ are similar.} \qquad \text{§ 326}$$

$$\therefore AB : A'B' = OA : O'A'. \qquad \text{§ 319}$$

Also $AB : A'B' = OM : O'M'$, § 328

(*the homologous altitudes of similar △s have the same ratio as their bases*).

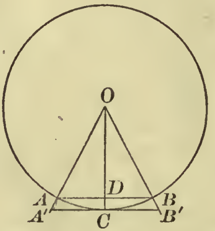
$$\therefore P : P' = OA : O'A' = OM : OM'.$$

Q. E. D.

414. COR. *The areas of two regular polygons of the same number of sides are to each other as the squares of the radii of their circumscribed circles, and also as the squares of the radii of their inscribed circles.* § 376

Proposition VI. Theorem.

415. *The difference between the lengths of the perimeters of a regular inscribed polygon and of a similar circumscribed polygon is indefinitely diminished as the number of the sides of the polygons is indefinitely increased.*



Let P and P' denote the lengths of the perimeters, AB and $A'B'$ two corresponding sides, OA and OA' the radii, of the polygons.

To prove that as the number of the sides of the polygons is indefinitely increased, $P'-P$ is indefinitely diminished.

Proof. Since the polygons are similar,

$$P' : P = OA' : OA. \qquad \S\ 413$$

By division, $P' - P : P = OA' - OA : OA.$

Whence $$P' - P = P \times \frac{OA' - OA}{OA}.$$

Draw the radius OC to the point of contact of $A'B'$.

In the $\triangle\ OA'C$, $\quad OA' - OC < A'C, \qquad \S\ 137$

(*the difference between two sides of a $\triangle$ is less than the third side*).

Substituting OA for its equal OC,

$$OA' - OA < A'C.$$

But as the number of sides of the polygon is indefinitely increased, the length of each side is indefinitely diminished; that is, $A'B'$, and consequently $A'C$, is indefinitely diminished.

Therefore $OA' - OA$, which is less than $A'C$, is indefinitely diminished; and the fraction $\frac{OA' - OA}{OA}$, the denominator of which is the constant OA, is indefinitely diminished.

But P always remains less than the circumference.

Therefore $P' - P$ is indefinitely diminished. Q. E. D.

416. Cor. *The difference between the areas of a regular inscribed polygon and of a similar circumscribed polygon is indefinitely diminished as the number of the sides of the polygons is indefinitely increased.*

For, if S and S' denote the areas of the polygons,

$$S' : S = (\overline{OA'}^2 : \overline{OA}^2) = \overline{OA'}^2 : \overline{OC}^2. \qquad \text{§ 414}$$

By division, $S' - S : S = \overline{OA'}^2 - \overline{OC}^2 : \overline{OC}^2$.

Whence $S' - S = S \times \frac{\overline{OA'}^2 - \overline{OC}^2}{\overline{OC}^2} = S \times \frac{\overline{A'C}^2}{\overline{OC}^2}$.

Since $A'C$ can be indefinitely diminished by increasing the number of the sides, $S' - S$ can be indefinitely diminished.

417. Scholium. The perimeter P' is constantly greater than P, and the area S' is constantly greater than S; for the radius OA' is constantly greater than OA. But P' constantly decreases and P constantly increases (§ 410), and the area S' constantly decreases, and the area S constantly increases, as the number of sides of the polygons is increased.

Since the difference between P' and P can be made as small as we please, but cannot be made absolutely zero, and since P' is decreasing while P is increasing, it is evident that P' and P tend towards a common limit. This common limit is *the length of the circumference.* § 259

Also, since the difference between the areas S' and S can be made as small as we please, but cannot be made absolutely zero, and since S' is decreasing, while S is increasing, it is evident that S' and S tend towards a common limit. This common limit is *the area of the circle.*

Proposition VII. Theorem.

418. *Two circumferences have the same ratio as their radii.*

Let C and C' be the circumferences, R and R' the radii, of the two circles Q and Q'.

To prove $C : C' = R : R'$.

Proof. Inscribe in the ⊙ two similar regular polygons, and denote their perimeters by P and P'.

Then $P : P' = R : R'$ (§ 413); that is, $R' \times P = R \times P'$.

Conceive the number of the sides of these similar regular polygons to be indefinitely increased, the polygons continuing to have an equal number of sides.

Then $R' \times P$ will continue equal to $R \times P'$, and P and P' will approach indefinitely C and C' as their respective limits.

$\therefore R' \times C = R \times C'$ (§ 260); that is, $C : C' = R : R'$.

Q. E. D.

419. Cor. *The ratio of the circumference of a circle to its diameter is constant.* For, in the above proportion, by doubling both terms of the ratio $R : R'$, we have

$$C : C' = 2R : 2R'.$$

By alternation, $C : 2R = C' : 2R'$.

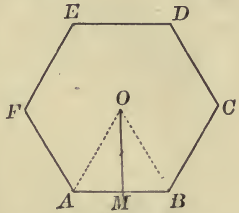
This constant ratio is denoted by π, so that for any circle whose diameter is $2R$ and circumference C, we have

$$\frac{C}{2R} = \pi, \text{ or } C = 2\pi R.$$

420. Scholium. The ratio π is incommensurable, and therefore can be expressed in figures only approximately.

Proposition VIII. Theorem.

421. *The area of a regular polygon is equal to one-half the product of its apothem by its perimeter.*



Let P represent the perimeter, R the apothem, and S the area of the regular polygon ABC, etc.

To prove $S = \frac{1}{2} R \times P$.

Proof. Draw OA, OB, OC, etc.

The polygon is divided into as many △ as it has sides.

The apothem is the common altitude of these △,

and the area of each △ is equal to $\frac{1}{2} R$ multiplied by the base. § 368

Hence the area of all the △ is equal to $\frac{1}{2} R$ multiplied by the sum of all the bases.

But the sum of the areas of all the △ is equal to the area of the polygon,

and the sum of all the bases of the △ is equal to the perimeter of the polygon.

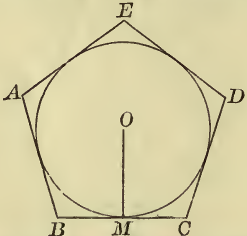
Therefore $S = \frac{1}{2} R \times P$.

Q. E. D.

422. In different circles *similar arcs*, *similar sectors*, and *similar segments* are such as correspond to equal angles at the centre.

Proposition IX. Theorem.

423. *The area of a circle is equal to one-half the product of its radius by its circumference.*



Let R represent the radius, C the circumference, and S the area, of the circle.

To prove $S = \frac{1}{2} R \times C$.

Proof. Circumscribe any regular polygon about the circle, and denote its perimeter by P.

Then the area of this polygon $= \frac{1}{2} R \times P$, § 421

Conceive the number of sides of the polygon to be indefinitely increased; then the perimeter of the polygon approaches the circumference of the circle as its limit, and the area of the polygon approaches the circle as its limit.

But the area of the polygon continues to be equal to one-half the product of the radius by the perimeter, however great the number of sides of the polygon.

Therefore $S = \frac{1}{2} R \times C$. § 260

Q. E. D.

424. Cor. 1. *The area of a sector equals one-half the product of its radius by its arc.* For the sector is such a part of the circle as its arc is of the circumference.

425. Cor. 2. *The area of a circle equals π times the square of its radius.*

For the area of the $\odot = \frac{1}{2} R \times C = \frac{1}{2} R \times 2\pi R = \pi R^2$.

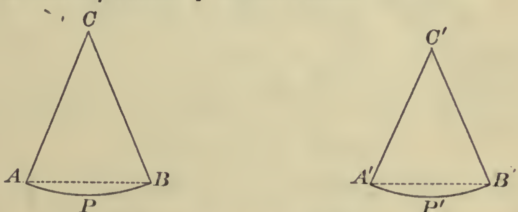
426. Cor. 3. *The areas of two circles are to each other as the squares of their radii.* For, if S and S' denote the areas, and R and R' the radii,

$$S : S' = \pi R^2 : \pi R'^2 = R^2 : R'^2.$$

427. Cor. 4. *Similar arcs, being like parts of their respective circumferences, are to each other as their radii; similar sectors, being like parts of their respective circles, are to each other as the squares of their radii.*

Proposition X. Theorem.

428. ***The areas of two similar segments are to each other as the squares of their radii.***



Let AC and $A'C'$ be the radii of the two similar segments ABP and $A'B'P'$.

To prove $ABP : A'B'P' = \overline{AC}^2 : \overline{A'C'}^2$.

Proof. The sectors ACB and $A'C'B'$ are similar, § 422
(having the ∠s at the centre, C and C', equal).

In the △s ACB and $A'C'B'$

$\angle C = \angle C'$, $AC = CB$, and $A'C' = C'B'$.

Therefore the △s ACB and $A'C'B'$ are similar. § 326

Now sector ACB : sector $A'C'B' = \overline{AC}^2 : \overline{A'C'}^2$, § 427

and $\triangle ACB : \triangle A'C'B' = \overline{AC}^2 : \overline{A'C'}^2$. § 375

Hence $\dfrac{\text{sector } ACB - \triangle ACB}{\text{sector } A'C'B' - \triangle A'C'B'} = \dfrac{\overline{AC}^2}{\overline{A'C'}^2}$. § 301

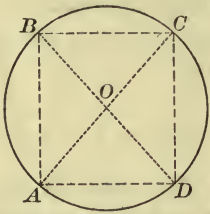
That is, $ABP : A'B'P' = \overline{AC}^2 : \overline{A'C'}^2$.

Q. E. D.

Problems of Construction.

Proposition XI. Problem.

429. *To inscribe a square in a given circle.*



Let O be the centre of the given circle.

To inscribe a square in the circle.

Construction. Draw the two diameters AC and $BD \perp$ to each other.

Join AB, BC, CD, and DA.

Then $ABCD$ is the square required.

Proof. The $\triangle$ ABC, BCD, etc., are rt. $\triangle$, § 264

(being inscribed in a semicircle),

and the sides AB, BC, etc., are equal, § 230

(in the same ⊙ equal arcs are subtended by equal chords).

Hence the figure $ABCD$ is a square. § 171

Q. E. F.

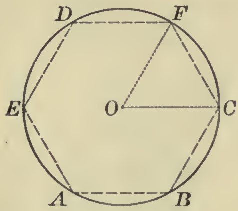
430. Cor. *By bisecting the arcs AB, BC, etc., a regular polygon of eight sides may be inscribed in the circle; and, by continuing the process, regular polygons of sixteen, thirty-two, sixty-four, etc., sides may be inscribed.*

Ex. 376. The area of a circumscribed square is equal to twice the area of the inscribed square.

Ex. 377. If the length of the side of an inscribed square is 2 inches, what is the length of the circumscribed square?

Proposition XII. Problem.

431. *To inscribe a regular hexagon in a given circle.*



Let O be the centre of the given circle.

To inscribe in the given circle a regular hexagon.

Construction. From O draw any radius, as OC.

From C as a centre, with a radius equal to OC, describe an arc intersecting the circumference at F.

Draw OF and CF.

Then CF is a side of the regular hexagon required.

Proof. The $\triangle\ OFC$ is equilateral and equiangular.

Hence the $\angle\ FOC$ is $\frac{1}{3}$ of 2 rt. $\angle$s, or $\frac{1}{6}$ of 4 rt. $\angle$s. § 138

And the arc FC is $\frac{1}{6}$ of the circumference $ABCF$.

Therefore the chord FC, which subtends the arc FC, is a side of a regular hexagon;

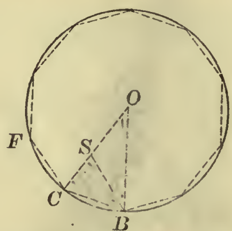
and the figure CFD, etc., formed by applying the radius six times as a chord, is a regular hexagon. Q. E. F.

432. Cor. 1. *By joining the alternate vertices A, C, D, an equilateral triangle is inscribed in the circle.*

433. Cor. 2. *By bisecting the arcs AB, BC, etc., a regular polygon of twelve sides may be inscribed in the circle; and, by continuing the process, regular polygons of twenty-four, forty-eight, etc., sides may be inscribed.*

PROPOSITION XIII. PROBLEM.

434. *To inscribe a regular decagon in a given circle.*



Let O be the centre of the given circle.

To inscribe a regular decagon in the given circle.

Construction. Draw the radius OC,

and divide it in extreme and mean ratio, so that OC shall be to OS as OS is to SC. § 355

From C as a centre, with a radius equal to OS,

describe an arc intersecting the circumference at B, and draw BC.

Then BC is a side of the regular decagon required.

Proof. Draw BS and BO.

By construction $OC : OS = OS : SC$,

and $BC = OS$.

$$\therefore\ OC : BC = BC : SC.$$

Moreover, the $\angle\ OCB = \angle\ SCB$. Iden.

Hence the $\triangle$ OCB and BCS are similar, § 326

(*having an $\angle$ of the one equal to an $\angle$ of the other, and the including sides proportional*).

But the $\triangle\ OCB$ is isosceles,

(*its sides OC and OB being radii of the same circle*).

$\therefore \triangle\ BCS$, which is similar to the $\triangle\ OCB$, is isosceles,

and $CB = BS = OS$.

$\therefore$ the $\triangle SOB$ is isosceles, and the $\angle O = \angle SBO$.

But the ext. $\angle CSB = \angle O + \angle SBO = 2\angle O$. § 145

Hence $\angle SCB\,(= \angle CSB) = 2\angle O$. § 154

and $\angle OBC\,(= \angle SCB) = 2\angle O$. § 154

$\therefore$ the sum of the $\angle$s of the $\triangle OCB = 5\angle O = 2$ rt. $\angle$s,

and $\angle O = \frac{1}{5}$ of 2 rt. $\angle$s, or $\frac{1}{10}$ of 4 rt. $\angle$s.

Therefore the arc BC is $\frac{1}{10}$ of the circumference,

and the chord BC is a side of a regular inscribed decagon.

Hence, to inscribe a regular decagon, divide the radius in extreme and mean ratio, and apply the greater segment ten times as a chord.

Q. E. F.

435. Cor. 1. *By joining the alternate vertices of a regular inscribed decagon, a regular pentagon is inscribed.*

436. Cor. 2. *By bisecting the arcs BC, CF, etc., a regular polygon of twenty sides may be inscribed; and, by continuing the process, regular polygons of forty, eighty, etc., sides may be inscribed.*

Let R denote the radius of a regular inscribed polygon, r the apothem, a one side, A an interior angle, and C the angle at the centre; show that

Ex. 378. In a regular inscribed triangle $a = R\sqrt{3}$, $r = \frac{1}{2}R$, $A = 60°$, $C = 120°$.

Ex. 379. In an inscribed square $a = R\sqrt{2}$, $r = \frac{1}{2}R\sqrt{2}$, $A = 90°$, $C = 90°$.

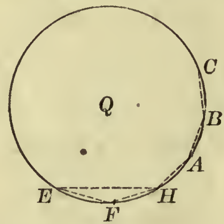
Ex. 380. In a regular inscribed hexagon $a = R$, $r = \frac{1}{2}R\sqrt{3}$, $A = 120°$ $C = 60°$.

Ex. 381. In a regular inscribed decagon

$$a = \frac{R(\sqrt{5}-1)}{2},\quad r = \tfrac{1}{4}R\sqrt{10+2\sqrt{5}},\quad A = 144°,\quad C = 36°.$$

Proposition XIV. Problem.

437. *To inscribe in a given circle a regular pentedecagon, or polygon of fifteen sides.*



Let Q be the given circle.

To inscribe in Q a regular pentedecagon.

Construction. Draw EH equal to a side of a regular inscribed hexagon, § 431

and EF equal to a side of a regular inscribed decagon. § 434

Join FH.

Then FH will be a side of a regular inscribed pentedecagon.

Proof. The arc EH is $\frac{1}{6}$ of the circumference,

and the arc EF is $\frac{1}{10}$ of the circumference.

Hence the arc FH is $\frac{1}{6}-\frac{1}{10}$, or $\frac{1}{15}$, of the circumference,

and the chord FH is a side of a regular inscribed pentedecagon.

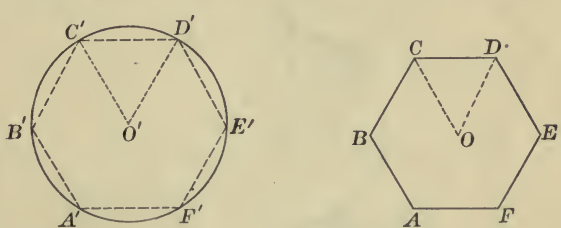
By applying FH fifteen times as a chord, we have the polygon required.

Q. E. F.

438. Cor. *By bisecting the arcs FH, HA, etc., a regular polygon of thirty sides may be inscribed; and, by continuing the process, regular polygons of sixty, one hundred and twenty, etc., sides, may be inscribed.*

Proposition XV. Problem.

439. *To inscribe in a given circle a regular polygon similar to a given regular polygon.*



Let ABCD, etc., be the given regular polygon, and C′D′E′ the given circle.

To inscribe in the circle a regular polygon similar to ABCD, etc.

Construction. From O, the centre of the given polygon,

draw OD and OC.

From O', the centre of the given circle,

draw $O'C'$ and $O'D'$,

making the $\angle O' = \angle O$.

Draw $C'D'$.

Then $C'D'$ will be a side of the regular polygon required.

Proof. Each polygon will have as many sides as the $\angle O$ $(= \angle O')$ is contained times in 4 rt. ∠s.

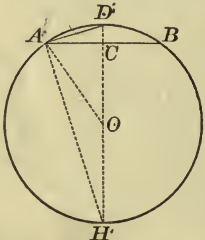
Therefore the polygon $C'D'E'$, etc., is similar to the polygon CDE, etc., § 411

(*two regular polygons of the same number of sides are similar*).

Q. E. F.

Proposition XVI. Problem.

440. *Given the radius and the side of a regular inscribed polygon, to find the side of the regular inscribed polygon of double the number of sides.*



Let AB be a side of the regular inscribed polygon.

To find the value of AD, a side of a regular inscribed polygon of double the number of sides.

From D draw DH through the centre O, and draw OA, AH.

DH is $\perp$ to AB at its middle point C. § 123

In the rt. $\triangle OAC$, $\overline{OC}^2 = \overline{OA}^2 - \overline{AC}^2$. § 339

That is, $OC = \sqrt{\overline{OA}^2 - \overline{AC}^2}$.

But $AC = \frac{1}{2} AB$; hence $\overline{AC}^2 = \frac{1}{4} \overline{AB}^2$.

Therefore, $OC = \sqrt{\overline{OA}^2 - \frac{1}{4}\overline{AB}^2}$.

In the rt. $\triangle DAH$, § 264

$$\overline{AD}^2 = DH \times DC \qquad \text{§ 334}$$

$$= 2\,OA\,(OA - OC),$$

and $$AD = \sqrt{2\,OA\,(OA - OC)}.$$

If we denote the radius by R, and substitute $\sqrt{R^2 - \frac{1}{4}\overline{AB}^2}$ for OC, then

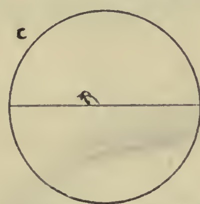
$$AD = \sqrt{2\,R\left(R - \sqrt{R^2 - \tfrac{1}{4}\overline{AB}^2}\right)}$$

$$= \sqrt{R\left(2\,R - \sqrt{4\,R^2 - \overline{AB}^2}\right)}.$$

Q. E. F.

PROPOSITION XVII. PROBLEM.

441. *To compute the ratio of the circumference of a circle to its diameter approximately.*



Let C be the circumference, and R the radius.

To find the numerical value of π.

$$2\pi R = \text{[illegible]} \qquad \S\ 419$$

Therefore when R [illegible], $\pi = \frac{1}{2}C$.

We make the following computations by the use of the formula obtained in the last proposition, when $R = 1$, and $AB = 1$ (a side of a regular hexagon).

No. Sides.	Form of Computation.	Length of Side.	Length of Perimeter
12	$c_1 = \sqrt{2 - \sqrt{4 - 1^2}}$	0.51763809	6.21165708
24	$c_2 = \sqrt{2 - \sqrt{4 - (0.51763809)^2}}$	0.26105238	6.26525722
48	$c_3 = \sqrt{2 - \sqrt{4 - (0.26105238)^2}}$	0.13080626	6.27870041
96	$c_4 = \sqrt{2 - \sqrt{4 - (0.13080626)^2}}$	0.06533817	6.28206396
192	$c_5 = \sqrt{2 - \sqrt{4 - (0.06543817)^2}}$	0.03272346	6.28290510
384	$c_6 = \sqrt{2 - \sqrt{4 - (0.03272346)^2}}$	0.01636228	6.28311544
768	$c_7 = \sqrt{2 - \sqrt{4 - (0.01636228)^2}}$	0.00818121	6.28316941

Hence we may consider 6.28317 as approximately the circumference of a ⊙ whose radius is unity.

Therefore $\pi = \frac{1}{2}(6.28317) = 3.14159$ nearly. Q. E. F.

442. SCHOLIUM. In practice, we generally take

$$\pi = 3.1416, \quad \frac{1}{\pi} = 0.31831.$$

Maxima and Minima. — Supplementary.

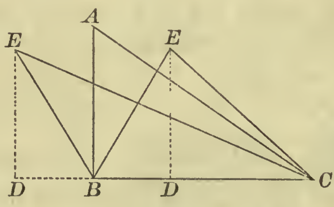
443. Among magnitudes of the same kind, that which is greatest is the *maximum*, and that which is smallest is the *minimum*.

Thus the diameter of a circle is the maximum among all inscribed straight lines; and a perpendicular is the minimum among all straight lines drawn from a point to a given line.

444. *Isoperimetric* figures are figures which have equal perimeters.

Proposition XVIII. Theorem.

445. *Of all triangles having two given sides, that in which these sides include a right angle is the maximum.*



Let the triangles ABC and EBC have the sides AB and BC equal respectively to EB and BC; and let the angle ABC be a right angle.

To prove $\triangle ABC > \triangle EBC$.

Proof. From E let fall the $\perp$ ED.

The $\triangle$s ABC and EBC, having the same base BC, are to each other as their altitudes AB and ED. § 370

Now $EB > ED$. § 114

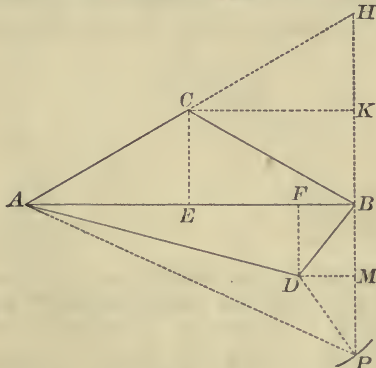
By hypothesis, $EB = AB$.

$\therefore AB > ED$.

$\therefore \triangle ABC > \triangle EBC$. Q. E. D.

PROPOSITION XIX. THEOREM.

446. *Of all triangles having the same base and equal perimeters, the isosceles triangle is the maximum.*



Let the $\triangle$ *ACB and ADB have equal perimeters, and let the* $\triangle ACB$ *be isosceles.*

To prove $\triangle ACB > \triangle ADB$.

Proof. Produce AC to H, making $CH = AC$, and join HB.

ABH is a right angle, for it will be inscribed in the semicircle whose centre is C, and radius CA.

Produce HB, and take $DP = DB$.

Draw CK and $DM \parallel$ to AB, and join AP.

Now $AH = AC + CB = AD + DB = AD + DP$.

But $AD + DP > AP$, hence $AH > AP$.

Therefore $HB > BP$. § 120

But $KB = \frac{1}{2} HB$ and $MB = \frac{1}{2} BP$. § 121

Hence $KB > MB$.

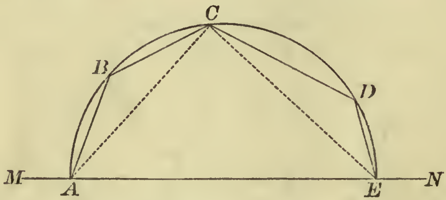
By § 180, $KB = CE$ and $MB = DF$, the altitudes of the $\triangle$ ACB and ADB.

Therefore $\triangle ABC > \triangle ADB$. § 370

Q. E. D

Proposition XX. Theorem.

447. *Of all polygons with sides all given but one, the maximum can be inscribed in a semicircle which has the undetermined side for its diameter.*



Let $ABCDE$ be the maximum of polygons with sides AB, BC, CD, DE, and the extremities A and E on the straight line MN.

To prove $ABCDE$ can be inscribed in a semicircle.

Proof. From *any* vertex, as C, draw CA and CE.

The $\triangle ACE$ must be the maximum of all $\triangle$ having the given sides CA and CE; otherwise, by increasing or diminishing the $\angle ACE$, keeping the sides CA and CE unchanged, but sliding the extremities A and E along the line MN, we can increase the $\triangle ACE$, while the rest of the polygon will remain unchanged, and therefore increase the polygon.

But this is contrary to the hypothesis that the polygon is the maximum polygon.

Hence the $\triangle ACE$ with the given sides CA and CE is the maximum.

Therefore the $\angle ACE$ is a right angle, § 445

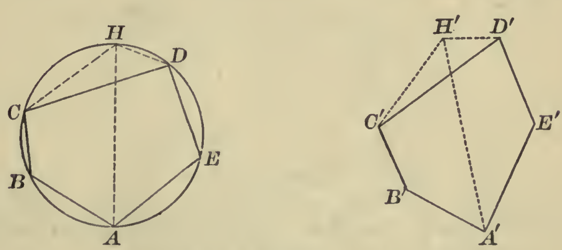
(*the maximum of $\triangle$ having two given sides is the $\triangle$ with the two given sides including a rt. $\angle$*).

Therefore C lies on the semi-circumference. § 264

Hence *every* vertex lies on the circumference; that is, the maximum polygon can be inscribed in a semicircle having the undetermined side for a diameter. Q. E. D.

Proposition XXI. Theorem.

448. *Of all polygons with given sides, that which can be inscribed in a circle is the maximum.*



Let $ABCDE$ be a polygon inscribed in a circle, and $A'B'C'D'E'$ be a polygon, equilateral with respect to $ABCDE$, which cannot be inscribed in a circle.

To prove $ABCDE$ *greater than* $A'B'C'D'E'$.

Proof. Draw the diameter AH.

Join CH and DH.

Upon $C'D'$ ($= CD$) construct the $\triangle C'H'D' = \triangle CHD$,

and draw $A'H'$.

Now $ABCH > A'B'C'H'$, § 447

and $AEDH > A'E'D'H'$,

(*of all polygons with sides all given but one, the maximum can be inscribed in a semicircle having the undetermined side for its diameter*).

Add these two inequalities, then

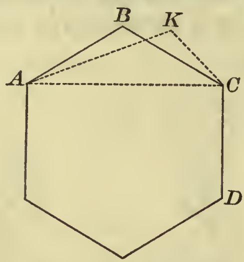
$$ABCHDE > A'B'C'H'D'E'.$$

Take away from the two figures the equal $\triangle$s CHD and $C'H'D'$.

Then $ABCDE > A'B'C'D'E'$. Q. E. D.

PROPOSITION XXII. THEOREM.

449. *Of isoperimetric polygons of the same number of sides, the maximum is equilateral.*



Let ABCD, etc., be the maximum of isoperimetric polygons of any given number of sides.

To prove AB, BC, CD, *etc., equal.*

Proof. Draw AC.

The $\triangle ABC$ must be the maximum of all the $\triangle$s which are formed upon AC with a perimeter equal to that of $\triangle ABC$.

Otherwise, a greater $\triangle AKC$ could be substituted for $\triangle ABC$, without changing the perimeter of the polygon.

But this is inconsistent with the hypothesis that the polygon $ABCD$, etc., is the maximum polygon.

$\therefore$ the $\triangle ABC$ is isosceles, § 446

(*of all $\triangle$s having the same base and equal perimeters, the isosceles $\triangle$ is the maximum*).

In like manner it may be proved that $BC = CD$, etc. Q. E. D.

450. COR. *The maximum of isoperimetric polygons of the same number of sides is a regular polygon.*

For, it is equilateral, § 449

(*the maximum of isoperimetric polygons of the same number of sides is equilateral*).

Also it can be inscribed in a circle, § 448

(*the maximum of all polygons formed of given sides can be inscribed in a* $\odot$).

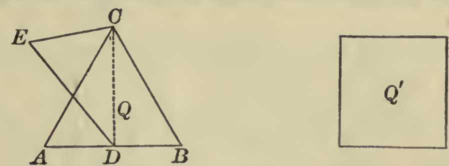
That is, it is equilateral and equiangular,

and therefore regular. § 395

Q. E. D.

PROPOSITION XXIII. THEOREM.

451. ***Of isoperimetric regular polygons, that which has the greatest number of sides is the maximum.***



Let Q be a regular polygon of three sides, and Q' a regular polygon of four sides, and let the two polygons have equal perimeters.

To prove Q' *greater than* Q.

Proof. Draw CD from C to any point in AB.

Invert the $\triangle\, CDA$ and place it in the position DCE, letting D fall at C, C at D, and A at E.

The polygon $DBCE$ is an irregular polygon of four sides, which by construction has the same perimeter as Q', and the same area as Q.

Then the irregular polygon $DBCE$ of four sides is less than the regular isoperimetric polygon Q' of four sides. § 450

In like manner it may be shown that Q' is less than a regular isoperimetric polygon of five sides, and so on. Q. E. D.

452. COR. *The area of a circle is greater than the area of any polygon of equal perimeter.*

---

382. Of all equivalent parallelograms having equal bases, the rectangle has the least perimeter.

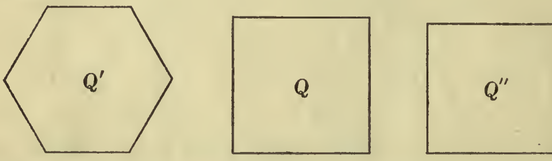
383. Of all rectangles of a given area, the square has the least perimeter.

384. Of all triangles upon the same base, and having the same altitude, the isosceles has the least perimeter.

385. To divide a straight line into two parts such that their product shall be a maximum.

Proposition XXIV. Theorem.

453. *Of regular polygons having a given area, that which has the greatest number of sides has the least perimeter.*



Let Q and Q' be regular polygons having the same area, and let Q' have the greater number of sides.

To prove the perimeter of Q greater than the perimeter of Q'.

Proof. Let Q'' be a regular polygon having the same perimeter as Q', and the same number of sides as Q.

Then $Q' > Q''$, § 451

(*of isoperimetric regular polygons, that which has the greatest number of sides is the maximum*).

But $Q = Q'$.

$\therefore Q > Q''$.

$\therefore$ the perimeter of $Q >$ the perimeter of Q''.

But the perimeter of $Q' =$ the perimeter of Q''. Cons.

$\therefore$ the perimeter of $Q >$ that of Q'.

Q. E. D.

454. Cor. *The circumference of a circle is less than the perimeter of any polygon of equal area.*

---

386. To inscribe in a semicircle a rectangle having a given area; a rectangle having the maximum area.

387. To find a point in a semi-circumference such that the sum of its distances from the extremities of the diameter shall be a maximum.

Theorems.

388. The side of a circumscribed equilateral triangle is equal to twice the side of the similar inscribed triangle. Find the ratio of their areas.

389. The apothem of an inscribed equilateral triangle is equal to half the radius of the circle.

390. The apothem of an inscribed regular hexagon is equal to half the side of the inscribed equilateral triangle.

391. The area of an inscribed regular hexagon is equal to three-fourths of that of the circumscribed regular hexagon.

392. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and the circumscribed equilateral triangles.

393. The area of an inscribed regular octagon is equal to that of a rectangle whose sides are equal to the sides of the inscribed and the circumscribed squares.

394. The area of an inscribed regular dodecagon is equal to three times the square of the radius.

395. Every equilateral polygon circumscribed about a circle is regular if it has an *odd* number of sides.

396. Every equiangular polygon inscribed in a circle is regular if it has an *odd* number of sides.

397. Every equiangular polygon circumscribed about a circle is regular.

398. Upon the six sides of a regular hexagon squares are constructed outwardly. Prove that the exterior vertices of these squares are the vertices of a regular dodecagon.

399. The alternate vertices of a regular hexagon are joined by straight lines. Prove that another regular hexagon is thereby formed. Find the ratio of the areas of the two hexagons.

400. The radius of an inscribed regular polygon is the mean proportional between its apothem and the radius of the similar circumscribed regular polygon.

401. The area of a circular ring is equal to that of a circle whose diameter is a chord of the outer circle and a tangent to the inner circle.

402. The square of the side of an inscribed regular pentagon is equal to the sum of the squares of the radius of the circle and the side of the inscribed regular decagon.

If R denotes the radius of a circle, and a one side of a regular inscribed polygon, show that:

403. In a regular pentagon, $a = \frac{R}{2}\sqrt{10 - 2\sqrt{5}}$.

404. In a regular octagon, $a = R\sqrt{2 - \sqrt{2}}$.

405. In a regular dodecagon, $a = R\sqrt{2 - \sqrt{3}}$.

406. If on the legs of a right triangle, as diameters, semicircles are described external to the triangle, and from the whole figure a semicircle on the hypotenuse is subtracted, the remainder is equivalent to the given triangle.

Numerical Exercises.

407. The radius of a circle $= r$. Find one side of the circumscribed equilateral triangle.

408. The radius of a circle $= r$. Find one side of the circumscribed regular hexagon.

409. If the radius of a circle is r, and the side of an inscribed regular polygon is a, show that the side of the similar circumscribed regular polygon is equal to
$$\frac{2ar}{\sqrt{4r^2 - a^2}}$$

410. The radius of a circle $= r$. Prove that the area of the inscribed regular octagon is equal to $2r^2\sqrt{2}$.

411. The sides of three regular octagons are 3 feet, 4 feet, and 5 feet, respectively. Find the side of a regular octagon equal in area to the sum of the areas of the three given octagons. $= \sqrt{50}$

412. What is the width of the ring between two concentric circumferences whose lengths are 440 feet and 330 feet? 17.5+

413. Find the angle subtended at the centre by an arc 6 feet 5 inches long, if the radius of the circle is 8 feet 2 inches. 45°+

414. Find the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle. 57.27°

415. What is the length of the arc subtended by one side of a regular dodecagon inscribed in a circle whose radius is 14 feet? 2.4434+ / Ft

416. Find the side of a square equivalent to a circle whose radius is 56 feet. 99.27 ft.

417. Find the area of a circle inscribed in a square containing 196 square feet. 153.9384 sq ft.

418. The diameter of a circular grass plot is 28 feet. Find the diameter of a circular plot just twice as large. 39.6- ft

419. Find the side of the largest square that can be cut out of a circular piece of wood whose radius is 1 foot 8 inches. 3.1+ ft.

420. The radius of a circle is 3 feet. What is the radius of a circle 25 times as large? $\frac{1}{5}$ as large? $\frac{1}{10}$ as large? = 37.1
15

421. The radius of a circle is 9 feet. What are the radii of the concentric circumferences that will divide the circle into three equivalent parts? 5.2- and 7.4-

422. The chord of half an arc is 12 feet, and the radius of the circle is 18 feet. Find the height of the arc.

423. The chord of an arc is 24 inches, and the height of the arc is 9 inches. Find the diameter of the circle.

424. Find the area of a sector, if the radius of the circle is 28 feet, and the angle at the centre $22\frac{1}{2}°$.

425. The radius of a circle $= r$. Find the area of the segment subtended by one side of the inscribed regular hexagon.

426. Three equal circles are described, each touching the other two. If the common radius is r, find the area contained between the circles.

Problems.

To circumscribe about a given circle:

427. An equilateral triangle.

428. A square.

429. A regular hexagon.

430. A regular octagon.

431. To draw through a given point a line so that it shall divide a given circumference into two parts having the ratio 3 : 7.

432. To construct a circumference equal to the sum of two given circumferences.

433. To construct a circle equivalent to the sum of two given circles.

434. To construct a circle equivalent to three times a given circle.

435. To construct a circle equivalent to three-fourths of a given circle.

To divide a given circle by a concentric circumference:

436. Into two equivalent parts.

437. Into five equivalent parts.

Miscellaneous Exercises.

Theorems.

438. The line joining the feet of the perpendiculars dropped from the extremities of the base of an isosceles triangle to the opposite sides is parallel to the base.

439. If AD bisect the angle A of a triangle ABC, and BD bisect the exterior angle CBF, then angle ADB equals one-half angle ACB.

440. The sum of the acute angles at the vertices of a pentagram (five-pointed star) is equal to two right angles.

441. The bisectors of the angles of a parallelogram form a rectangle.

442. The altitudes AD, BE, CF of the triangle ABC bisect the angles of the triangle DEF.

Hint. Circles with AB, BC, AC as diameters will pass through E and D, E and F, D and F, respectively.

443. The portions of any straight line intercepted between the circumferences of two concentric circles are equal.

444. Two circles are tangent internally at P, and a chord AB of the larger circle touches the smaller circle at C. Prove that PC bisects the angle APB.

Hint. Draw a common tangent at P, and apply §§ 263, 269, 145.

445. The diagonals of a trapezoid divide each other into segments which are proportional.

446. The perpendiculars from two vertices of a triangle upon the opposite sides divide each other into segments reciprocally proportional.

447. If through a point P in the circumference of a circle two chords are drawn, the chords and the segments between P and a chord parallel to the tangent at P are reciprocally proportional.

448. The perpendicular from any point of a circumference upon a chord is a mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.

449. In an isosceles right triangle either leg is a mean proportional between the hypotenuse and the perpendicular upon it from the vertex of the right angle.

450. The area of a triangle is equal to half the product of its perimeter by the radius of the inscribed circle.

451. The perimeter of a triangle is to one side as the perpendicular from the opposite vertex is to the radius of the inscribed circle.

452. The sum of the perpendiculars from any point within a convex equilateral polygon upon the sides is constant.

453. A diameter of a circle is divided into any two parts, and upon these parts as diameters semi-circumferences are described on opposite sides of the given diameter. Prove that the sum of their lengths is equal to the semi-circumference of the given circle, and that they divide the circle into two parts whose areas have the same ratio as the two parts into which the diameter is divided.

454. Lines drawn from one vertex of a parallelogram to the middle points of the opposite sides trisect one of the diagonals.

455. If two circles intersect in the points A and B, and through A any secant CAD is drawn limited by the circumferences at C and D, the straight lines BC, BD, are to each other as the diameters of the circles.

456. If three straight lines AA', BB', CC', drawn from the vertices of a triangle ABC to the opposite sides, pass through a common point O within the triangle, then

$$\frac{OA'}{AA'}+\frac{OB'}{BB'}+\frac{OC'}{CC'}=1.$$

457. Two diagonals of a regular pentagon, not drawn from a common vertex, divide each other in extreme and mean ratio.

Loci.

458. Find the locus of a point P whose distances from two given points A and B are in a given ratio $(m:n)$.

459. OP is any straight line drawn from a fixed point O to the circumference of a fixed circle; in OP a point Q is taken such that $OQ:OP$ is constant. Find the locus of Q.

460. From a fixed point A a straight line AB is drawn to any point in a given straight line CD, and then divided at P in a given ratio $(m:n)$. Find the locus of the point P.

461. Find the locus of a point whose distances from two given straight lines are in a given ratio. (The locus consists of two straight lines.)

462. Find the locus of a point the sum of whose distances from two given straight lines is equal to a given length k. (See Ex. 73.)

Problems.

463. Given the perimeters of a regular inscribed and a similar circumscribed polygon, to compute the perimeters of the regular inscribed and circumscribed polygons of double the number of sides.

464. To draw a tangent to a given circle such that the segment intercepted between the point of contact and a given straight line shall have a given length.

465. To draw a straight line equidistant from three given points.

466. To inscribe a straight line of given length between two given circumferences and parallel to a given straight line. (See Ex. 137.)

467. To draw through a given point a straight line so that its distances from two other given points shall be in a given ratio ($m : n$).

Hint. Divide the line joining the two other points in the given ratio.

468. Construct a square equivalent to the sum of a given triangle and a given parallelogram.

469. Construct a rectangle having the difference of its base and altitude equal to a given line, and its area equivalent to the sum of a given triangle and a given pentagon.

470. Construct a pentagon similar to a given pentagon and equivalent to a given trapezoid.

471. To find a point whose distances from three given straight lines shall be as the numbers m, n, and p. (See Ex. 461.)

472. Given two circles intersecting at the point A. To draw through A a secant BAC such that AB shall be to AC in a given ratio ($m : n$).

Hint. Divide the line of centres in the given ratio.

473. To construct a triangle, given its angles and its area.

474. To construct an equilateral triangle having a given area.

475. To divide a given triangle into two equal parts by a line drawn parallel to one of the sides.

476. Given three points A, B, C. To find a fourth point P such that the areas of the triangles APB, APC, BPC, shall be equal.

477. To construct a triangle, given its base, the ratio of the other sides, and the angle included by them.

478. To divide a given circle into any number of equivalent parts by concentric circumferences.

479. In a given equilateral triangle, to inscribe three equal circles tangent to each other and to the sides of the triangle.

SOLID GEOMETRY.

BOOK VI.

LINES AND PLANES IN SPACE.

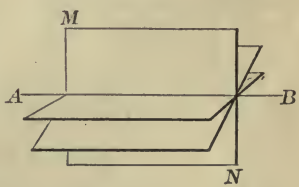
DEFINITIONS.

455. A *plane* has already been defined as a surface such that a straight line joining any two points in it lies wholly in the surface.

A plane is considered to be indefinite in extent, so that however far the straight line is produced, all its points lie in the plane; but a plane is usually represented by a quadrilateral supposed to lie in the plane.

456. A plane is said to be *determined* by lines or points, if no other plane can contain these lines or points without being coincident with that plane.

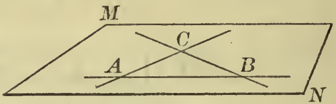
457. A plane can be made to turn about any straight line in it as an axis, and be made to assume as many different positions as we choose. Hence it is evident that a plane is not determined by a straight line.



In making a complete revolution about a straight line as an axis the plane passes successively through all points of space.

458. *A plane is determined by a straight line and a point without that line.*

If a plane containing the straight line AB revolve about this line as an axis until it contains the point C, the plane is determined. For if the plane revolve either way about the line AB as an axis, it will cease to contain the point C.



459. *Three points not in a straight line determine a plane.*

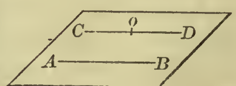
For, by joining any two of the points we have a straight line and a point without it, and these determine a plane. § 458

460. *Two intersecting straight lines determine a plane.*

For, a plane containing one of these straight lines and any point of the other line in addition to the point of intersection is determined. § 458

461. *Two parallel straight lines determine a plane.*

For, two parallel straight lines lie in the same plane, and a plane containing either of these parallels and any point in the other is determined. § 458



462. A straight line is *perpendicular to a plane* if it is perpendicular to every straight line of the plane drawn through its foot; that is, through the point where it meets the plane.

In this case the plane is perpendicular to the line.

463. A line is *oblique to a plane* if it is not perpendicular to all straight lines drawn in the plane through its foot.

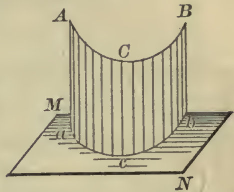
464. The *distance* from a point to a plane is the perpendicular distance from the point to the plane.

465. A line is *parallel to a plane* if it cannot meet the plane however far both are produced.

In this case the plane is parallel to the line.

466. *Two planes are parallel* if they cannot meet however far they are produced.

467. The *projection of a point* on a plane is the foot of the perpendicular from the point to the plane.



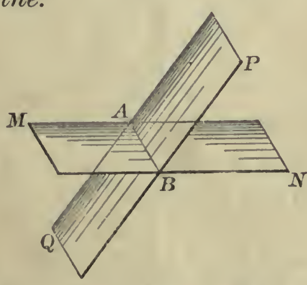
468. The *projection of a line* on a plane is the locus of the projections of all its points.

469. The *angle* which a line makes with a plane is the angle which it makes with its projection on the plane.

470. The *intersection of two planes* is the locus of all the points common to the two planes.

Proposition I. Theorem.

471. *If two planes cut each other, their intersection is a straight line.*



Let MN and PQ be two planes which cut one another.

To prove their intersection a straight line.

Proof. Let A and B be two points common to the two planes.

Draw a straight line through the points A and B.

Since the points A and B are common to the two planes, this straight line lies in both planes. § 455

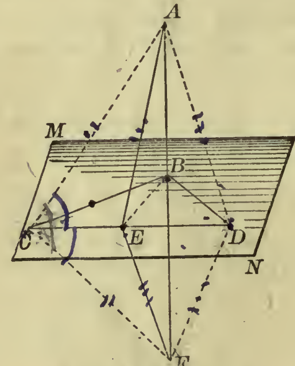
No point out of this line can be in both planes; for only *one* plane can contain a straight line and a point without the line.

Therefore the straight line through A and B is the locus of all the points common to the two planes, and is consequently the intersection of the planes (§ 470). Q. E. D.

Perpendicular Lines and Planes.

Proposition II. Theorem.

472. *If a straight line is perpendicular to each of two other straight lines at their point of intersection, it is perpendicular to the plane of the two lines.*



Let AB be perpendicular to BC and BD at B.

To prove AB perpendicular to the plane MN of these lines.

Proof. Through B draw in MN any other straight line BE, and draw CD cutting BC, BE, BD, at C, E, and D.

Prolong AB to F, making $BF = AB$, and join A and F to each of the points C, E, and D.

Since BC and BD are each $\perp$ to AF at its middle point,

$\therefore AC = FC$ and $AD = FD$. § 122

$\therefore \triangle ACD = \triangle FCD$ (§ 160), and hence $\angle ACD = \angle FCD$.

Now in the $\triangle$s ACE and FCE

$AC = FC$, $CE = CE$, and $\angle ACE = \angle FCE$.

$\therefore \triangle ACE = \triangle FCE$ (§ 150), and hence $AE = FE$.

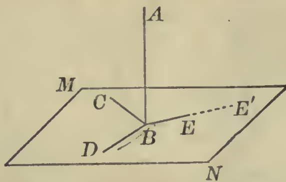
$\therefore BE$ is $\perp$ to AF at B. § 123

Hence AB is $\perp$ to BE, *any*, that is, *every*, straight line drawn in MN through B, and therefore is $\perp$ to MN. § 462

Q. E. D.

PROPOSITION III. THEOREM.

473. *Every perpendicular that can be drawn to a straight line at a given point lies in a plane perpendicular to the line at the given point.*



Let the plane MN be perpendicular to AB at B.

To prove that BE, any perpendicular to AB at B, lies in the plane MN.

Proof. Let the plane containing AB and BE intersect MN in the line BE'; then AB is $\perp$ to BE'. § 462

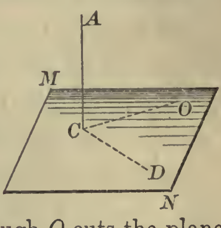
Since in the plane ABE only one $\perp$ can be drawn to AB at B (§ 89), BE and BE' coincide, and BE lies in MN.

Hence every $\perp$ to AB at B lies in the plane MN.

Q. E. D.

474. COR. 1. *At a given point in a straight line one plane perpendicular to the line can be drawn, and only one.*

475. COR. 2. *Through a given point without a straight line, one plane can be drawn perpendicular to the line, and only one.*



Let AC be the line, and O the point without it. In the plane OCA draw OC $\perp$ to AC, and in another plane containing AC draw CD $\perp$ to AC at C. Then CO and CD determine a plane $\perp$ to AC.

Every plane $\perp$ to AC and passing through O cuts the plane OCA in a line $\perp$ to AC and containing O. This $\perp$ coincides, then, with OC, and every such plane is $\perp$ to AC at C. But only one plane can be $\perp$ to AC at C (§ 474). Hence only one plane can be drawn from O $\perp$ to AC at C.

PROPOSITION IV. THEOREM.

476. *Through a given point one perpendicular can be drawn to a given plane, and only one.*

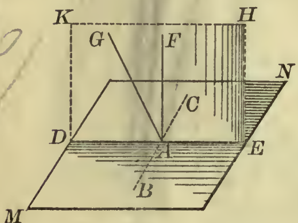


FIG. 1.

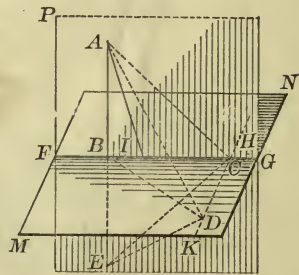


FIG. 2.

CASE I. *When the given point is in the given plane.*

Let A be the given point in the plane MN (Fig. 1).

To prove that one perpendicular can be erected to the plane MN at A, and only one.

Proof. Draw in MN any line BC through A, and pass through A a plane $DEHK \perp$ to BC, and cutting MN in DE.

At A erect in the plane $DEHK$ a line $AF \perp$ to DE.

The line BC, being $\perp$ to the plane $DEHK$ by construction, is $\perp$ to AF which passes through its foot in the plane. § 462

That is, AF is $\perp$ to BC; and as it is $\perp$ to DE by construction, it is $\perp$ to the plane MN. § 472

Moreover, every other line AG drawn from A, is oblique to MN. For AF and AG intersecting in A determine a plane $DEHK$, which cuts MN in the straight line DE; and as AF is $\perp$ to MN, it is $\perp$ to DE (§ 462); hence AG is oblique to DE (§ 89), and therefore to MN (§ 463).

Therefore AF is the only $\perp$ to MN at the point A.

CASE II. *When the given point is without the given plane.*

Let A be the given point, and MN the plane.

To prove that one perpendicular can be drawn from A to MN, and only one.

Proof. Draw in MN any line HK, and pass through A a plane $PQ \perp$ to HK, cutting MN in FG, and HK in C.

Let fall from A, in the plane PQ, a $\perp$ AB upon FG.

Draw in the plane MN any other line BD from B.

Prolong AB to E, making $BE = AB$,

and join A and E to each of the points C and D.

Since DC is $\perp$ to PQ by construction, and CA and CE lie in PQ, the $\angle$s DCA and DCE are right angles. § 462

In the rt. $\triangle$s DCA and DCE,

DC is common, and $CA = CE$. § 122

$\therefore \triangle DCA = \triangle DCE$ (§ 151), and hence $DA = DE$.

$\therefore BD$ is $\perp$ to AE at B. § 123

That is, AB is $\perp$ to BD, any straight line drawn in MN through its foot, and therefore $\perp$ to MN.

Moreover, every other straight line AI, drawn from A, is oblique to MN. For the lines AB and AI determine a plane PG which cuts the plane MN in the line FG. The line AB being $\perp$ to the plane MN, is $\perp$ to FG (§ 462). Therefore AI is oblique to FG, and consequently to MN (§ 463).

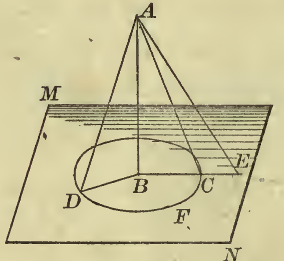
Therefore AB is the only $\perp$ from A to MN.

Q. E. D.

477. COR. *The perpendicular is the shortest line from a point to a plane, for it is the shortest line from the point to any straight line of the plane passing through its foot* (§ 114).

PROPOSITION V. THEOREM.

478. *Oblique lines drawn from a point to a plane, and meeting the plane at equal distances from the foot of the perpendicular, are equal; and of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular the more remote is the greater.*



Let AC and AD cut off the equal distances BC and BD from the foot of the perpendicular AB, and let AD and AE cut off the unequal distances BD and BE, and BE be greater than BD.

To prove $AC = AD$, *and* $AE > AD$.

Proof. The right $\triangle$ ABC and ABD have AB common, and $BC = BD$ by hypothesis.

Therefore they are equal, and $AC = AD$.

The right $\triangle$ ABE, ABC have AB common, and $BE > BC$.

Therefore $AE > AC$ (§ 119), and hence $AE > AD$.

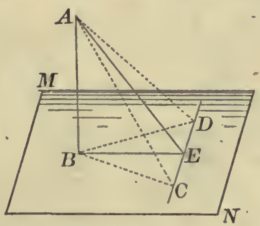
Q. E. D.

479. COR. 1. *Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

480. COR. 2. *The locus of a point in space equidistant from all points in the circumference of a circle is a straight line passing through the centre and perpendicular to the plane of the circle.*

Proposition VI. Theorem.

481. *If from the foot of a perpendicular to a plane a straight line is drawn at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to any point of the perpendicular is perpendicular to the line of the plane.*



Let AB be a perpendicular to the plane MN, BE a perpendicular from B to any line CD in MN, and EA a line from E to any point A in AB.

To prove AE *perpendicular to* CD.

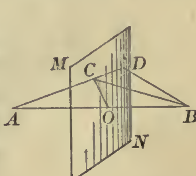
Proof. Take EC and ED equal; draw BC, BD, AC, AD.

Now $BC = BD$ (§ 116), and $AC = AD$ (§ 478).

$\therefore AE$ is $\perp$ to CD, § 123

Q. E. D.

482. Cor. *The locus of a point in space equidistant from the extremities of a straight line is the plane perpendicular to this line at its middle point.*

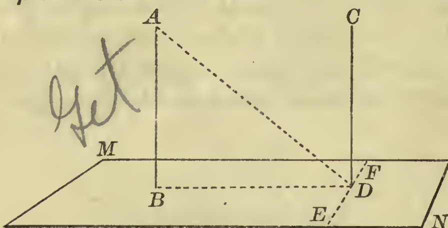


For, if the plane MN is $\perp$ to AB at its middle point O, and any point C in this plane is joined to A, O, and B, CO is $\perp$ to AB; therefore CA and CB are equal. § 116

Also, since all the $\perp$s to the line AB at the point O lie in the plane MN (§ 473), any point D without the plane MN cannot lie in a $\perp$ to AB at O, and therefore is unequally distant from A and B. § 122

Proposition VII. Theorem.

483. *Two straight lines perpendicular to the same plane are parallel.*



Let AB **and** CD **be perpendicular to** MN.

To prove AB *and* CD *parallel.*

Proof. Let A be any point in AB; join AD and BD, and through D draw EF in the plane $MN \perp$ to BD.

Then CD is $\perp$ to EF. § 462

Also, AD is $\perp$ to EF. § 481

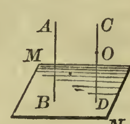
Therefore CD, AD, and BD, being $\perp$ to EF at D, lie in the same plane. § 473

Therefore AB and CD lie in the same plane; and since, by hypothesis, they are $\perp$ to MN, they are $\perp$ to BD. § 462

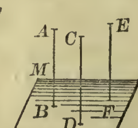
Therefore AB and CD are parallel. Q. E. D.

484. Cor. 1. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*

For, if AB and CD are $\parallel$, and AB is $\perp$ to the plane MN, and if through any point O of CD a line is drawn $\perp$ to MN, it will be $\parallel$ to AB (§ 483). Since through the point O only one line can be drawn $\parallel$ to AB (§ 101), CD will coincide with this $\perp$ and be $\perp$ to MN.



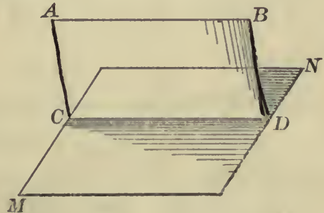
485. Cor. 2. *If two straight lines AB and EF are parallel to a third line CD, they are parallel to each other.* For, a plane $MN \perp$ to CD, is $\perp$ to AB and EF (§ 484). Hence AB and EF, being $\perp$ to MN, are parallel (§ 483).



Parallel Lines and Planes.

Proposition VIII. Theorem.

486. *If two straight lines are parallel, every plane containing one of the lines, and only one, is parallel to the other line.*



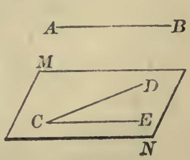
***Let** AB **and** CD **be two parallel lines, and** MN **any plane containing** CD **and not** AB.*

To prove AB and MN parallel.

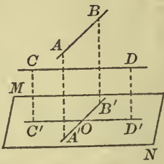
Proof. The lines AB and CD are in the same plane $ABCD$, which intersects the plane MN in the line CD.

Since AB is in the plane AD, it must meet the plane MN, if at all, in a point common to the two planes; that is, in a point of their intersection CD. But since AB is $\parallel$ to CD, it cannot meet CD. Therefore AB cannot meet the plane MN, and hence is $\parallel$ to MN. Q. E. D.

487. Cor. 1. *Through a given straight line a plane can be passed parallel to any other given straight line in space.* For, if a plane is passed through one of the lines AB and any point C of the other line CD, and a line CE is drawn in this plane $\parallel$ to AB, the plane MN determined by CD and CE is $\parallel$ to AB. § 486



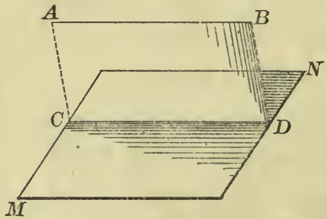
488. Cor. 2. *Through a given point a plane can be passed parallel to any two given straight lines in space.*



For, if O is the given point, and AB and CD the given lines, by drawing through O a line $A'B'$ ∥ to AB in the plane determined by AB and O, and also a line $C'D'$ ∥ to CD in the plane determined by CD and O, we shall have two lines $A'B'$ and $C'D'$ which determine a plane passing through O and ∥ to each of the lines AB and CD. § 486

Proposition IX. Theorem.

489. *If a given straight line is parallel to a given plane, the intersection of the given plane with any plane passed through the given line is parallel to that line.*



Let the line AB be parallel to the plane MN, and let CD be the intersection of MN with any plane passed through AB.

To prove AB and CD parallel.

Proof. The lines AB and CD are in the same plane $ABCD$, and therefore if the line AB meets the line CD, it must meet the plane MN.

But AB is by hypothesis ∥ to MN, and therefore cannot meet it; that is, it cannot meet CD, however far they may be produced.

Hence AB and CD are parallel.

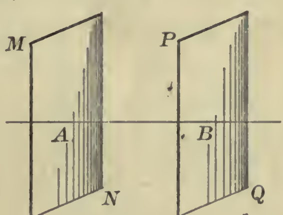
Q. E. D.

490. Cor. *If a given straight line and a plane are parallel, a parallel to the given line drawn through any point of the plane lies in the plane.*

For the plane determined by the given line AB and any point C of the plane cuts MN in a line $CD \parallel$ to AB (§ 489); but through C only one parallel to AB can be drawn (§ 101); therefore a line drawn through $C \parallel$ to AB coincides with CD, and hence lies in the plane MN.

Proposition X. Theorem.

491. *Two planes perpendicular to the same straight line are parallel.*



***Let** MN **and** PQ **be two planes perpendicular to the straight line** AB.*

To prove MN and PQ parallel.

Proof. MN and PQ cannot meet. For if they could meet, we should have two planes from a point of their intersection $\perp$ to the same straight line. But this is impossible, § 475

(*through a given point without a straight line, only one plane can be passed $\perp$ to the given line*).

Therefore MN and PQ are parallel.

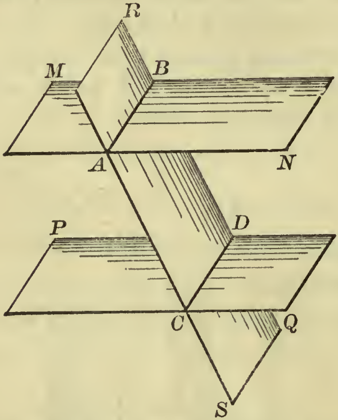
Q. E. D.

Ex. 480. Find the locus of a point in space equidistant from two given parallel planes.

Ex. 481. Find the locus of a point in space equidistant from two given points and also equidistant from two given parallel planes.

Proposition XI. Theorem.

492. *The intersections of two parallel planes by a third plane are parallel lines.*



Let the parallel planes *MN* and *PQ* be cut by *RS*.

To prove the intersections AB and CD parallel.

Proof. *AB* and *CD* are in the same plane *RS*.

They are also in the parallel planes *MN* and *PQ*, which cannot meet however far they extend.

Therefore *AB* and *CD* cannot meet, and are parallel.

Q. E. D.

493. Cor. 1. *Parallel lines included between parallel planes are equal.*

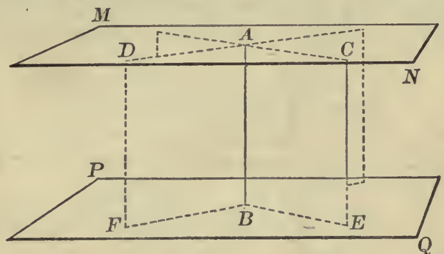
For, if the lines *AC* and *BD* are ∥, the plane of these lines will intersect *MN* and *PQ* in the ∥ lines *AB* and *CD* (§ 492). Hence *ABDC* is a parallelogram, and *AC* and *BD* are equal.

494. Cor. 2. *Two parallel planes are everywhere equally distant.*

For ⊥s dropped from *any* points in *MN* to *PQ* measure the distances of these points from *PQ*. But these ⊥s are parallel (§ 483), and hence equal (§ 493). Therefore *all* points in *MN* are equidistant from *PQ*.

Proposition XII. Theorem.

495. *A straight line perpendicular to one of two parallel planes is perpendicular to the other.*



Let AB be perpendicular and PQ parallel to MN.

To prove that AB is perpendicular to PQ.

Proof. Pass through the line AB any two planes intersecting MN in the lines AC and AD, and PQ in BE and BF. Then AC and AD are $\parallel$ to BE and BF respectively. § 492

But AB is $\perp$ to AC and AD (§ 462), and is therefore $\perp$ to their parallels BE and BF. § 102

Therefore, AB is $\perp$ to PQ. § 472

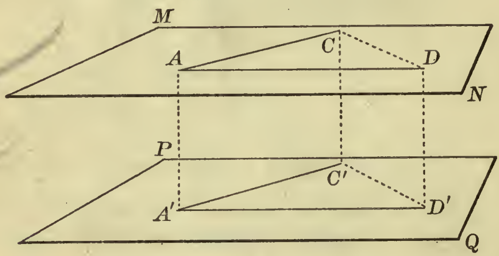
Q. E. D.

496. Cor. 1. *Through a given point A one plane, and only one, can be drawn parallel to a given plane PQ.* For, if a line is drawn from A $\perp$ to PQ, a plane passing through A $\perp$ to this line is $\parallel$ to PQ (§ 491); and since only one plane can be drawn through a point $\perp$ to a given line (§ 474), only one plane can be drawn through A $\parallel$ to PQ.

497. Cor. 2. *If two intersecting lines AC and AD are each parallel to a plane PQ, the plane of these lines MN is parallel to PQ.* For draw AB $\perp$ to PQ, and through the point B draw BE and BF $\parallel$ to AC and AD. Then BE and BF lie in the plane PQ (§ 490). Hence AB is $\perp$ to BE and BF. Therefore AB is $\perp$ to AC and AD (§ 102), and hence to the plane MN (§ 472). Hence MN and PQ are parallel. § 491

Proposition XIII. Theorem.

498. *If two angles not in the same plane have their sides respectively parallel and lying in the same direction, they are equal, and their planes are parallel.*



Let the angles A and A' be respectively in the planes MN and PQ and have AD parallel to $A'D'$ and AC parallel to $A'C$ and lying in the same direction.

To prove $\angle A = \angle A'$, and $MN \parallel$ to PQ.

Proof. Take AD and $A'D'$ equal, also AC and $A'C'$ equal.

Join AA', DD', CC', CD, $C'D'$.

Since AD is equal and $\parallel$ to $A'D'$, the figure $ADD'A'$ is a parallelogram, and $AA' = DD'$. § 182

In like manner $AA' = CC'$.

Also, since CC' and DD' are each $\parallel$ to AA', and equal to AA', they are $\parallel$ and equal.

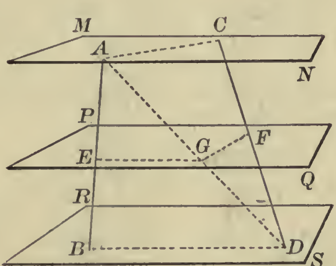
Therefore $CDD'C'$ is a parallelogram, and $CD = C'D'$.

$\therefore \triangle ADC = \triangle A'D'C'$, and $\angle A = \angle A'$. § 160

Also, since PQ is $\parallel$ to each of the lines AC and AD (§ 486), PQ is $\parallel$ to the plane of these lines MN (§ 497). Q. E. D.

Proposition XIV. Theorem.

499. *If two straight lines are intersected by three parallel planes, their corresponding segments are proportional.*



Let AB and CD be intersected by the parallel planes MN, PQ, RS, in the points A, E, B, and C, F, D.

To prove $AE : EB = CF : FD$.

Proof. Draw AD cutting the plane PQ in G.

Join EG and FG.

Then EG is ∥ to BD, and GF is ∥ to AC. § 492

$\therefore AE : EB = AG : GD$, § 309

and $CF : FD = AG : GD$.

$\therefore AE : EB = CF : FD$.

Q. E. D.

Ex. 482. The line AB meets three parallel planes in the points A, E, B; and the line CD meets the same planes in the points C, F, D. If $AE = 6$ inches, $BE = 8$ inches, $CD = 12$ inches, compute CF and FD.

Ex. 483. To draw a perpendicular to a given plane from a given point without it.

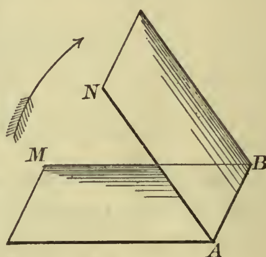
Ex. 484. To erect a perpendicular to a given plane at a given point in it.

Dihedral Angles.

500. The *opening* between two intersecting planes is called a *dihedral angle.*

The line of intersection AB of the planes is the *edge*, and the planes MA and NB are the *faces* of the dihedral angle.

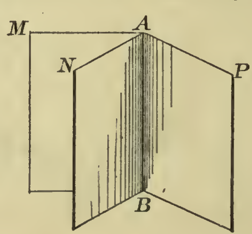
501. A dihedral angle is designated by its edge, or by its two faces and its edge. Thus, the dihedral angle in the margin may be designated by AB, or by $M\text{-}AB\text{-}N$.



502. In order to have a clear notion of the *magnitude* of the dihedral angle AB, suppose a plane at first in coincidence with MA to turn about the edge AB, in the direction indicated by the arrow, until it coincides with the face NB. The *amount of rotation* of this plane is the dihedral angle AB.

503. Two dihedral angles are *equal* when they can be made to coincide.

504. Two dihedral angles $M\text{-}AB\text{-}N$ and $N\text{-}AB\text{-}P$ are *adjacent* if they have a common edge AB, and a common face NAB, between them.



505. When a plane meets another plane and makes the *adjacent dihedral angles equal*, each of these angles is called a *right dihedral angle.*

506. A plane is *perpendicular* to another plane if it forms with this second plane a right dihedral angle.

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507. Two *vertical dihedral angles* are angles that have the same edge and the faces of the one are the prolongations of the faces of the other.

508. Dihedral angles are *acute*, *obtuse*, *complementary*, *supplementary*, under the same conditions as plane angles are acute, obtuse, complementary, supplementary, respectively.

509. The demonstrations of many properties of dihedral angles are identically the same as the demonstrations of analogous theorems of plane angles.

The following are examples:

1. If a plane meets another plane, it forms with it two adjacent dihedral angles whose sum is equal to two right dihedral angles.

2. If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are in the same plane.

3. If two planes intersect each other, their vertical dihedral angles are equal.

4. If a plane intersects two parallel planes, the exterior-interior dihedral angles are equal; the alternate-interior dihedral angles are equal; the two interior dihedral angles on the same side of the secant plane are supplements of each other.

5. When two planes are cut by a third plane, if the exterior-interior dihedral angles are equal, or the alternate-interior dihedral angles are equal, and the edges of the dihedral angles thus formed are parallel, the two planes are parallel.

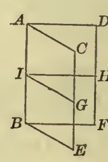
6. Two dihedral angles whose faces are parallel each to each are either equal or supplementary.

7. Two dihedral angles whose faces are perpendicular each to each are either equal or supplementary.

MEASURE OF DIHEDRAL ANGLES.

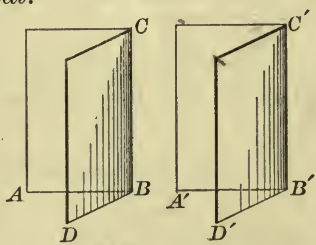
510. The *plane angle* of a dihedral angle is the plane angle formed by two straight lines, one in each plane, perpendicular to the edge at the same point.

511. *The plane angle of a dihedral angle has the same magnitude from whatever point in the edge the perpendiculars are drawn.* For any two such angles, as CAD, GIH, have their sides respectively parallel (§ 100), and hence are equal (§ 498).



PROPOSITION XV. THEOREM.

512. *Two dihedral angles are equal if their plane angles are equal.*



Let the two plane angles ABD and $A'B'D'$ of the two dihedral angles CB and $C'B'$ be equal.

To prove the dihedral angles CB and $C'B'$ equal.

Proof. Apply $B'C'$ to BC, making the plane angle $A'B'D'$ coincide with its equal ABD.

The line $B'C'$ being $\perp$ to the plane $A'B'D'$ will likewise be $\perp$ to the plane ABD at B, and take the direction BC, since at B only one $\perp$ can be erected to this plane. § 476

The two planes $A'B'C'$ and ABC, having in common two intersecting lines AB and BC, coincide. § 460

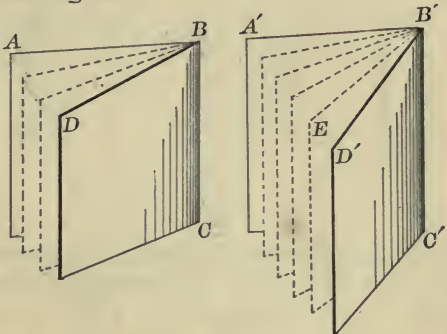
In like manner the planes $D'B'C'$ and DBC coincide.

Therefore the two dihedral angles coincide and are equal.

Q. E. D.

PROPOSITION XVI. THEOREM.

513. *Two dihedral angles have the same ratio as their plane angles.*



CASE I. *When the plane angles are commensurable.*

Let** A-BC-D **and** A′-B′C′-D′ **be two dihedral angles, and let their plane angles** ABD **and** A′B′D′ **be commensurable.

To prove $A\text{-}BC\text{-}D : A'\text{-}B'C'\text{-}D' = \angle ABD : \angle A'B'D'$.

Proof. Suppose the $\measuredangle$ ABD and $A'B'D'$ have a common measure, which is contained three times in $\angle ABD$ and five times in $\angle A'B'D'$.

Then $$\angle ABD : \angle A'B'D' = 3 : 5.$$

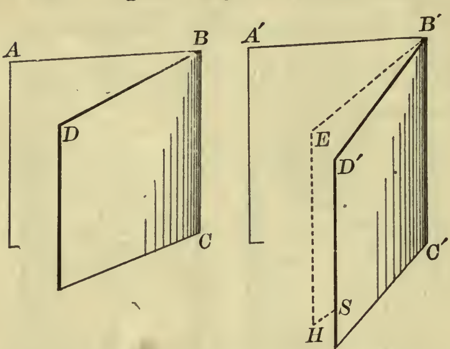
Apply this measure to $\angle ABD$ and $\angle A'B'D'$, and through the lines of division and the edges BC and $B'C'$ pass planes.

These planes divide $A\text{-}BC\text{-}D$ into three parts, and $A'\text{-}B'C'D'$ into five parts, all equal because they have equal plane angles.

Therefore $$A\text{-}BC\text{-}D : A'\text{-}B'C'\text{-}D' = 3 : 5.$$

Therefore $A\text{-}BC\text{-}D : A'\text{-}B'C'\text{-}D' = \angle ABD : \angle A'B'D'$.

CASE II. *When the plane angles are incommensurable.*



Let A-BC-D, A′-B′C-D′ be dihedral angles, and let their plane angles ABD, A′B′D′ be incommensurable.

To prove $A\text{-}BC\text{-}D : A'\text{-}B'C'\text{-}D' = \angle ABD : \angle A'B'D'$.

Proof. Divide the $\angle ABD$ into any number of equal parts, and apply one of these parts to the $\angle A'B'D'$ as a measure.

Since ABD and $A'B'D'$ are incommensurable, a certain number of these parts will form the $\angle A'B'E$, leaving a remainder $EB'D'$, less than one of the parts.

Pass a plane through $B'E$ and $B'C'$.

Since the plane angles of the dihedral angles $A\text{-}BC\text{-}D$ and $A'\text{-}B'C'\text{-}E$ are commensurable, we have by Case I.,

$$A\text{-}BC\text{-}D : A'\text{-}B'C'\text{-}E = \angle ABD : \angle A'B'E.$$

If the unit of measure is indefinitely diminished, these ratios continue equal, and approach indefinitely the limiting ratios,

$$A\text{-}BC\text{-}D : A'\text{-}B'C'\text{-}D, \text{ and } \angle ABD : \angle A'B'D'.$$

$$\therefore A\text{-}BC\text{-}D : A'\text{-}B'C'\text{-}D' = \angle ABD : \angle A'B'D'. \quad \S\ 260$$

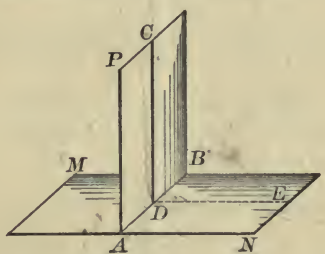
Q. E. D.

514. SCHOLIUM. The plane angle is taken as the *measure* of the dihedral angle. (Compare § 262.)

Planes Perpendicular to Each Other.

Proposition XVII. Theorem.

515. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Let the plane PAB be perpendicular to MN, and let CD be drawn in PAB perpendicular to their intersection AB.

To prove CD perpendicular to MN.

Proof. In the plane MN draw $DE \perp$ to AB at D.

Then CDE is the plane angle of the right dihedral angle P-AB-N, and is therefore a right angle.

By construction CDA is a right angle.

Therefore CD is $\perp$ to DA and DE at their point of intersection, and consequently $\perp$ to their plane MN. § 472

Q. E. D.

516. Cor. 1. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection will lie in the other.*

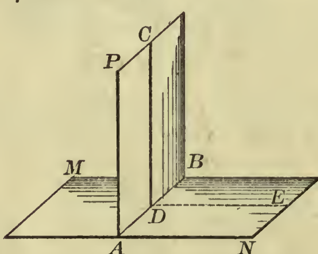
For, a line CD drawn in the plane $PAB \perp$ to AB at the point D will be $\perp$ to MN (§ 515). But at the point D only one $\perp$ can be drawn to MN (§ 476). Therefore a $\perp$ to MN erected at D will coincide with CD and lie in the plane PAB.

517. Cor. 2. *If two planes are perpendicular to each other, a perpendicular to one of them from any point of the other will lie in the other.*

For, a line CD drawn in the plane PAB from the point C $\perp$ to AB will be $\perp$ to MN (§ 515). But from the point C only one $\perp$ can be drawn to MN (§ 476). Therefore a $\perp$ to MN drawn from C will coincide with CD and lie in PAB.

Proposition XVIII. Theorem.

518. *If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to the first plane.*



Let CD be perpendicular to MN, and PAB be any plane passed through CD intersecting MN in AB.

To prove the plane PAB perpendicular to the plane MN.

Proof. Draw DE in the plane MN, and $\perp$ to AB.

Since CD is $\perp$ to MN, it is $\perp$ to AB.

Therefore $\angle\ CDE$ is the plane angle of $P\text{-}AB\text{-}N$.

But $\angle\ CDE$ is a right angle,

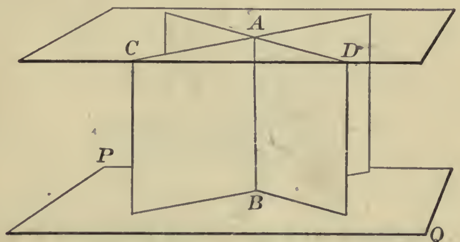
and therefore PAB is $\perp$ to MN. § 514

Q. E. D.

519. Cor. *A plane perpendicular to the edge of a dihedral angle is perpendicular to each of its faces.*

PROPOSITION XIX. THEOREM.

520. *If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.*



Let the planes *BD* and *BC* intersecting in the line *AB* be perpendicular to the plane *PQ*.

To prove AB perpendicular to the plane PQ.

Proof. A $\perp$ erected to PQ at B, a point common to the three planes, will lie in the two planes BC and BD. § 516

And since this $\perp$ lies in both the planes BC and BD, it must coincide with their intersection AB.

$\therefore AB$ is $\perp$ to the plane PQ.

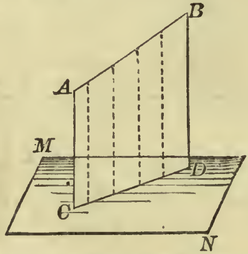
Q. E. D.

521. COR. 1. *If a plane PQ is perpendicular to each of two intersecting planes ABC and ABD, it is perpendicular to their intersection AB.*

522. COR. 2. *If a plane PQ is perpendicular to two planes ABC and ABD, which include a right dihedral angle, the intersection of any two of these planes is perpendicular to the third plane, and each of the three intersections is perpendicular to the other two.*

Proposition XX. Theorem.

523. *Through a given straight line not perpendicular to a plane, one plane, and only one, can be passed perpendicular to the given plane.*



Let AB be the given line not perpendicular to the plane MN.

To prove that one plane can be passed through AB perpendicular to MN, and only one.

Proof. From any point A of AB draw $AC \perp$ to MN,

and through AB and AC pass a plane AD.

The plane AD is $\perp$ to MN, since it passes through AC, a line $\perp$ to MN. § 518

Moreover, if two planes could be passed through $AB \perp$ to the plane MN, their intersection AB would be $\perp$ to MN. § 520

But this is impossible, since AB is by hypothesis oblique to the plane MN.

Hence through AB only one plane can be passed $\perp$ to MN.

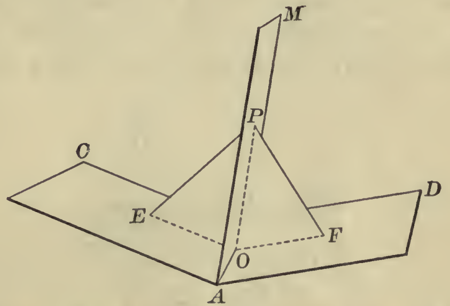
Q. E. D.

V **524.** Cor. *If a straight line is oblique to a plane, its projection is a straight line.*

For, the plane passed through $AB \perp$ to MN contains all the $\perp$s let fall from different points of AB upon MN (§ 516). Therefore the intersection CD of these planes is the locus of the projections of the points in AB. But the intersection CD is a straight line; that is, the projection of AB is a straight line.

PROPOSITION XXI. THEOREM.

525. *Every point in a plane which bisects a dihedral angle is equidistant from the faces of the angle.*



Let plane AM bisect the dihedral angle formed by the planes AD and AC; and let PE and PF be perpendiculars drawn from any point P in the plane AM to the planes AC and AD.

To prove $PE = PF$.

Proof. Through PE and PF pass a plane intersecting the planes AC and AD in the lines OE and OF, and join PO.

The plane PEF is $\perp$ to AC and to AD. § 518

Hence the plane PEF is $\perp$ to their intersection AO. § 521

$$\therefore \angle POE = \angle POF,$$

(*being measures respectively of the equal dihedral* $\angle$s *M-OA-C and M-OA-D*).

$$\therefore \text{rt. } \triangle POE = \text{rt. } \triangle POF. \quad \S\ 148$$

$$\therefore PE = PF.$$

Q. E. D.

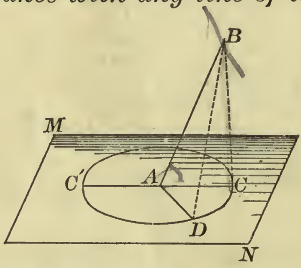
Ex. 485. Find the locus of a point in space equidistant from three given points not in a straight line.

Ex. 486. Given two points A and B on the same side of a given plane MN; find a point in this plane such that the sum of its distances from A and B shall be a minimum.

ANGLE OF A STRAIGHT LINE AND A PLANE.

PROPOSITION XXII. THEOREM.

526. *The acute angle which a straight line makes with its own projection upon a plane is the least angle which it makes with any line of the plane.*



Let BA meet the plane MN at A, and let AC be its projection upon the plane MN, and AD any other line drawn through A in the plane.

To prove $\angle BAC$ *less than* $\angle BAD$.

Proof. Take $AD = AC$, and join BD.

In the $\triangle$ BAC and BAD

$BA = BA$, $AC = AD$, but $BC < BD$. § 477

$\therefore \angle BAC$ is less than $\angle BAD$, § 153

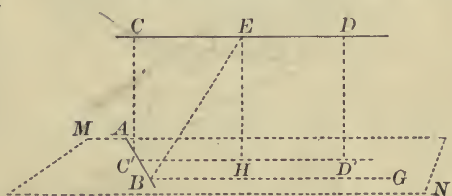
Q. E. D.

527. SCHOLIUM. If the straight line AC turns about the point A, the angle BAC increases; it is a right angle when AC is perpendicular to its initial position; then it becomes obtuse, and reaches its maximum value when AC falls upon AC' the prolongation of CA. Afterwards the angle passes through the same values in reverse order.

A Perpendicular between Two Straight Lines.

Proposition XXIII. Theorem.

528. *Between two straight lines not in the same plane, one common perpendicular can be drawn, and only one.*



Let AB and CD be the given lines.

To prove that one common perpendicular can be drawn between them, and only one.

Proof. Through any point B of AB draw $BG \parallel$ to CD, and let MN be the plane determined by AB and BG.

Through CD pass the plane $CD' \perp$ to MN, and intersecting AB at C'.

At C' erect a $\perp$ $C'C$ to the plane MN. $C'C$ will lie in the plane $C'D$ (§ 516), and be $\perp$ to AB and $C'D'$ (§ 462).

Since $C'C$ is $\perp$ to $C'D'$, it is $\perp$ to CD (§ 102).

Hence CC' is a common perpendicular to CD and AB.

Moreover, CC' is the only common perpendicular.

For, if any other line EB could be $\perp$ to CD and AB, it would be $\perp$ to BG and AB (§ 102), and hence $\perp$ to MN.

But EH in the plane CD' and $\perp$ to $C'D'$, is $\perp$ to MN (§ 515), and we should have two $\perp$s from E to MN.

But this is impossible. § 476

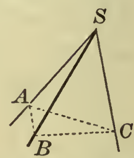
Hence CC' is the only common $\perp$ to CD and AB.

Q. E. D.

POLYHEDRAL ANGLES.

529. A *polyhedral angle* is the opening of three or more planes which meet at a common point.

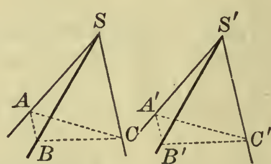
530. The common point S is the *vertex* of the angle, and the intersections of the planes SA, SB, etc., are its *edges;* the portions of the planes included between the edges are its *faces*, and the angles formed by the edges are its *face angles.*



531. The *magnitude* of a polyhedral angle depends upon the *relative position* of its faces, and not upon their extent.

532. In a polyhedral angle, every two adjacent edges form a face angle, and every two adjacent faces form a dihedral angle. These face angles and dihedral angles are the *parts* of the polyhedral angle.

533. Two polyhedral angles can be made to coincide and are *equal* if their corresponding parts are equal and arranged in the *same order.*



534. A polyhedral angle is *convex* if any section made by a plane cutting all its faces is a convex polygon.

535. A polyhedral angle is called *trihedral*, *tetrahedral*, etc., according as it has *three* faces, *four* faces, etc.

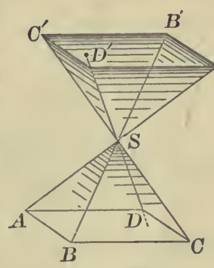
536. A trihedral angle is called *rectangular*, *bi-rectangular*, *tri-rectangular*, according as it has one, two, or three right dihedral angles.

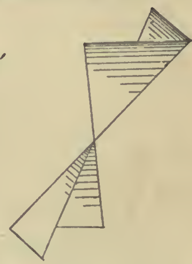
Two adjacent walls and the floor of a rectangular room form a tri-rectangular trihedral angle.

537. A trihedral angle is called *isosceles* if it has two of its face angles equal.

Symmetrical Polyhedral Angles.

538. If the edges of a given polyhedral angle $S\text{-}ABCD$ are produced through the vertex S, another polyhedral angle $S\text{-}A'B'C'D'$ is formed, *symmetrical* with respect to $S\text{-}ABCD$. The face angles ASB, BSC, etc., are equal respectively to the face angles $A'SB'$, $B'SC'$, etc., since they are vertical angles.



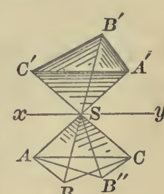


Also the dihedral angles whose edges are SA, SB, etc., are equal respectively to the dihedral angles whose edges are SA', SB', etc., since they are vertical dihedral angles. (The second figure shows a pair of vertical dihedral angles.)

The edges of $S\text{-}ABCD$ are arranged from *left to right* in the order SB, SC, SD, but the edges of $S\text{-}A'B'C'D'$ are arranged from *right to left* in the order SB', SC', SD'; that is, in an order the reverse of the order of the edges in $S\text{-}ABCD$.

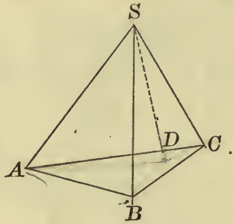
Two symmetrical polyhedral angles, therefore, have all their parts equal, each to each, but *arranged in reverse order*.

In general, two symmetrical polyhedral angles are not superposable. Thus, if the trihedral angle $S\text{-}A'B'C'$ is made to turn 180° about the bisector xy of the angle $A'SC$, the side SA' will coincide with SC, SC' with SA, and the face $A'SC'$ with ASC; but the dihedral angle SA, and hence the dihedral angle SA', not being equal to SC, the plane $A'SB'$ will not coincide with BSC; and, for a similar reason, the plane $C'SB'$ will not coincide with ASB. Hence the edge SB' takes some position SB'' not coincident with SB; that is, the trihedral angles are not superposable.



PROPOSITION XXIV. THEOREM.

539. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



In the trihedral angle $S\text{-}ABC$ ***let the angle*** ASC ***be greater than*** ASB ***or*** BSC.

To prove $\angle ASB + \angle BSC$ *greater than* $\angle ASC$.

Proof. In ASC draw SD, making $\angle ASD = \angle ASB$.

Through any point D of SD draw ADC in the plane ASC.

Take $SB = SD$.

Pass a plane through the line AC and the point B.

In the $\triangle$ ASD and ASB,

$AS = AS$, $SD = SB$, and $\angle ASD = \angle ASB$.

$\therefore \triangle ASD = \triangle ASB$ (§ 150), and $AD = AB$.

In the $\triangle ABC$, $\quad AB + BC > AC$ § 137

But $\quad AB \quad = AD$

By subtraction, $\quad BC > DC$

In the $\triangle$ BSC and DSC,

$SC = SC$, and $SB = SD$, but $BC > DC$.

Therefore $\angle BSC$ is greater than $\angle DSC$. § 153

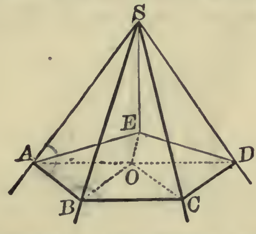
$\therefore \angle$s $ASB + BSC$ are greater than $\angle$s $ASD + DSC$.

That is, $\angle$s $ASB + BSC$ are greater than $\angle ASC$.

Q. E. D.

Proposition XXV. Theorem.

540. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Let S be a convex polyhedral angle, and let its faces be cut by a plane, making the section $ABCDE$ a convex polygon.

To prove $\angle ASB + \angle BSC$, *etc., less than four rt.* $\angle$s.

Proof. From any point O within the polygon draw OA, OB, OC, OD, OE.

The number of the $\triangle$s having their common vertex at O will be the same as the number having their common vertex at S.

Therefore the sum of all the $\angle$s of the $\triangle$s having the common vertex at S is equal to the sum of all the $\angle$s of the $\triangle$s having the common vertex at O.

But in the trihedral $\angle$s formed at A, B, C, etc.

$\angle SAE + \angle SAB$ is greater than $\angle EAB$,

and $\angle SBA + \angle SBC$ is greater than $\angle ABC$, etc. § 539

Hence the sum of the $\angle$s at the bases of the $\triangle$s whose common vertex is S is greater than the sum of the $\angle$s at the bases of the $\triangle$s whose common vertex is O.

Therefore the sum of the $\angle$s at the vertex S is less than the sum of the $\angle$s at the vertex O.

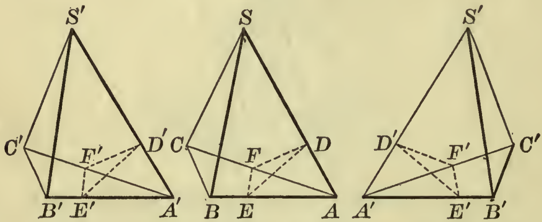
But the sum of the $\angle$s at $O = 4$ rt. $\angle$s. § 92

Therefore the sum of the $\angle$s at S is less than 4 rt. $\angle$s.

Q. E. D.

PROPOSITION XXVI. THEOREM.

541. *Two trihedral angles are equal or symmetrical when the three face angles of the one are respectively equal to the three face angles of the other.*



In the trihedral angles S and S' let the angles ASB ASC, BSC, be equal to the angles $A'S'B'$, $A'S'C'$, $B'S'C'$, respectively.

To prove S-ABC and S'-$A'B'C'$ equal or symmetrical.

Proof. On the edges of these angles take the six equal distances SA, SB, SC, $S'A'$, $S'B'$, $S'C'$.

Draw AB, BC, AC, $A'B'$, $B'C'$, $A'C'$.

The isosceles $\triangle$s SAB, SAC, SBC, are equal respectively to $S'A'B'$, $S'A'C'$, $S'B'C'$. § 150

AB, AC, BC are equal respectively to $A'B'$, $A'C'$, $B'C'$.

$$\therefore \triangle ABC = \triangle A'B'C'. \qquad \S\ 160$$

At any point D in SA draw DE and DF in the faces ASB and ASC respectively, and $\perp$ to SA.

These lines meet AB and AC respectively,

(*since the* $\angle$s *SAB and SAC are acute, each being one of the equal* $\angle$s *of an isosceles* $\triangle$).

Join EF.

On $S'A'$ take $A'D' = AD$.

Draw $D'E'$ and $D'F'$ in the faces of $A'S'B'$ and $A'S'C'$ respectively, $\perp$ to $S'A'$, and join $E'F'$.

In the rt. $\triangle$ ADE and $A'D'E'$,

$$AD = A'D', \text{ and } \angle DAE = \angle D'A'E'.$$

$$\therefore \text{rt. } \triangle ADE = \text{rt. } \triangle A'D'E'. \qquad \S\ 149$$

$$\therefore AE = A'E' \text{ and } DE = D'E'.$$

In like manner we may prove $AF = A'F'$ and $DF = D'F'$.

Hence in the $\triangle$ AEF and $A'E'F'$

$$AE = A'E', AF = A'F', \text{ and } \angle EAF = \angle E'A'F'.$$

$$\therefore \triangle AEF = \triangle A'E'F', \text{ and } EF = E'F', \qquad \S\ 150$$

Hence, in the $\triangle$ EDF and $E'D'F'$ we have

$$ED = E'D', DF = D'F', \text{ and } EF = E'F'.$$

$$\therefore \triangle EDF = \triangle E'D'F' \text{ and } \angle EDF = \angle E'D'F'. \quad \S\ 160$$

Therefore the dihedral angle $B\text{-}AS\text{-}C$ equals dihedral angle $B'\text{-}A'S'\text{-}C'$,

(*since* $\angle$ *EDF and E'D'F', the measures of these dihedral* $\angle$*, are equal*).

In like manner it may be proved that the dihedral angles $A\text{-}BS\text{-}C$ and $A\text{-}CS\text{-}B$ are equal respectively to the dihedral angles $A'\text{-}B'S'\text{-}C'$ and $A'\text{-}C'S'\text{-}B'$. Q. E. D.

This demonstration applies to either of the two figures denoted by $S'\text{-}A'B'C'$, which are symmetrical with respect to each other. If the first of these figures is taken, S and S' are equal. If the second is taken, S and S' are symmetrical.

542. SCHOLIUM. *If two trihedral angles have three face angles of the one equal to three face angles of the other, then the dihedral angles of the one are respectively equal to the dihedral angles of the other.*

Ex. 487. An isosceles trihedral angle and its symmetrical trihedral angle are superposable.

Ex. 488. Find the locus of a point equidistant from the three edges of a trihedral angle.

Ex. 489. Find the locus of a point equidistant from the three faces of a trihedral angle.

BOOK VII.

POLYHEDRONS, CYLINDERS, AND CONES.

POLYHEDRONS.

543. A *polyhedron* is a solid bounded by planes.

The bounding planes, limited by each other, are the *faces*, their intersections are the *edges*, and the intersections of the edges are the *vertices*, of the polyhedron.

544. A *diagonal* of a polyhedron is a straight line joining any two vertices not in the same face.

545. A polyhedron of four faces is called a *tetrahedron;* one of six faces, a *hexahedron;* one of eight faces, an *octahedron;* one of twelve faces, a *dodecahedron;* one of twenty faces, an *icosahedron.*

546. A polyhedron is *convex* if the section made by any plane cutting it is a convex polygon.

Only convex polyhedrons are considered in this work.

547. The *volume* of a solid is its numerical measure, referred to another solid taken as the *unit of volume.*

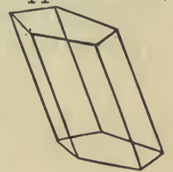
548. A polyhedron of six faces, each face a square, is called a *cube;* and the cube whose edge is the linear unit is generally taken as the unit of volume.

549. Two solids are *equivalent* if their volumes are equal.

550. Two polygons are *parallel* if their sides are respectively parallel.

Prisms and Parallelopipeds.

551. A *prism* is a polyhedron of which two opposite faces, called *bases*, are parallel polygons, and the other faces, called *lateral faces*, intersect in parallel lines, called *lateral edges*. The lateral edges are equal (§493), the lateral faces are parallelograms (168), and the bases are equal (§§179, 498).

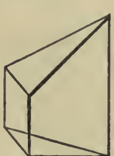


552. The sum of the areas of the lateral faces of a prism is called its *lateral area*.

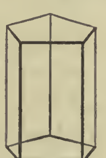
553. The *altitude* of a prism is the length of the perpendicular between the planes of its bases.

554. Prisms are called *triangular*, *quadrangular*, etc., according as their bases are triangles, quadrilaterals, etc.

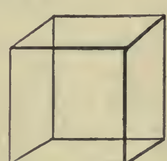
555. A *right section* of a prism is a section made by a plane perpendicular to its lateral edges.



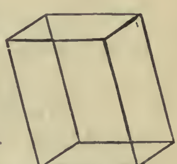
Truncated Prism.



Right Prism.



Rectangular Parallelopiped.



Parallelopiped.

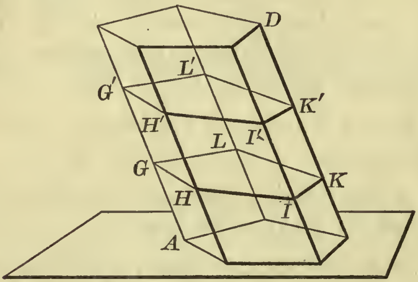
556. A *truncated prism* is the part of a prism included between the base and a section made by a plane inclined to the base and cutting all the lateral edges.

557. An *oblique prism* is a prism whose lateral edges are not perpendicular to its bases; a *right prism* is a prism whose lateral edges are perpendicular to its bases; a *regular prism* is a right prism whose bases are regular polygons.

558. A prism whose bases are parallelograms is called a *parallelopiped*. If its lateral edges are perpendicular to the bases, it is called a *right parallelopiped*. If its six faces are all rectangles, it is called a *rectangular parallelopiped*.

PROPOSITION I. THEOREM.

559. *The sections of a prism made by parallel planes are equal polygons.*



Let the prism AD be intersected by the parallel planes GK, G'K'.

To prove $GHIKL = G'H'I'K'L'$.

Proof. Since the intersections of two parallel planes by a third plane are parallel (§ 492), the sides GH, HI, IK, etc., are parallel respectively to the sides $G'H'$, $H'I'$, $I'K'$, etc.

The sides GH, HI, IK, etc., are equal respectively to $G'H'$, $H'I'$, $I'K'$, etc.,

since parallel lines comprehended between parallel lines are equal. § 180

The $\angle$s GHI, HIK, etc., are equal respectively to $\angle$s $G'H'I'$, $H'I'K'$, etc.,

since two $\angle$s not in the same plane, having their sides respectively parallel and lying in the same direction, are equal. § 498

Therefore $GHIKL = G'H'I'K'L'$, § 203

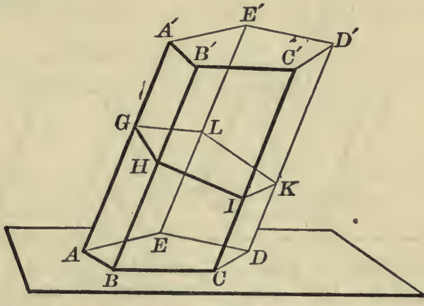
because they are mutually equiangular and equilateral.

Q. E. D.

560. COR. *Any section of a prism parallel to the base is equal to the base; and all right sections of a prism are equal.*

Proposition II. Theorem.

561. *The lateral area of a prism is equal to the product of a lateral edge by the perimeter of the right section.*



Let $GHIKL$ be a right section of the prism AD'.

To prove lateral area of $AD' = AA'(GH + HI +$ etc.).

Proof. Consider the lateral edges AA', BB', etc., to be the bases of the ▱s AB', BC', etc., which form the lateral surface of the prism.

Then the bases of these ▱s are all equal. § 551

Since the sides of the right section, GH, HI, etc., are ⊥ to AA', BB', etc. (§ 462), they are the altitudes of these ▱s, and the sum of the altitudes GH, HI, IK, etc., is the perimeter of the right section.

The area of each ▱ is the product of its base and altitude. § 365

Hence, the sum of the areas of the ▱s is the product of a lateral edge AA' by the perimeter of the right section.

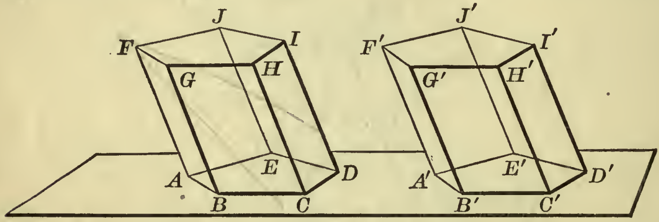
But the sum of the areas of the ▱s is the lateral area of the prism.

Therefore the lateral area of the prism is equal to the product of a lateral edge by the perimeter of a right section. Q. E. D.

562. Cor. *The lateral area of a right prism is equal to the altitude multiplied by the perimeter of the base.*

PROPOSITION III. THEOREM.

563. *Two prisms are equal if three faces including a trihedral angle of the one are respectively equal to three faces including a trihedral angle of the other, and are similarly placed.*



In the prisms AI and $A'I'$, let AD, AG, AJ, be respectively equal to $A'D'$, $A'G'$, $A'J'$, and similarly placed.

To prove prism AI = prism $A'I'$.

Proof. By hypothesis the face $\angle s$ BAE, BAF, EAF, are equal to the face $\angle s$ $B'A'E'$, $B'A'F'$, $E'A'F'$, respectively.

Therefore the trihedral angle $A = A'$. § 541

Apply A to its equal A'; then the faces AD, AG, AJ, will coincide with the equal faces $A'D'$, $A'G'$, $A'J'$, respectively, the points C and D falling at C' and D'.

As the lateral edges of the prisms are parallel, CH will take the direction of $C'H'$, and DI of $D'I'$.

Since the points F, G, and J coincide with F', G', and J', each to each, the planes of the upper bases will coincide.

Hence H will coincide with H', and I with I'.

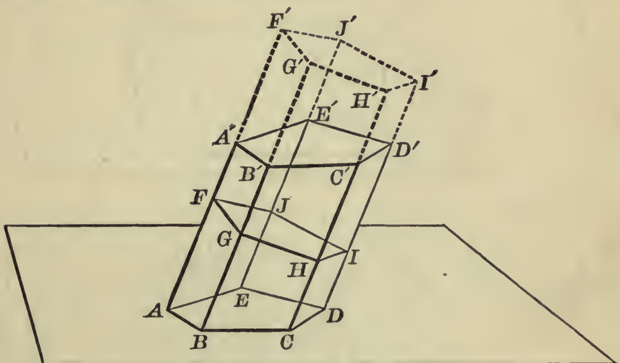
Therefore the prisms coincide and are equal. Q. E. D.

564. COR. 1. *Two truncated prisms are equal if three faces including a trihedral of the one are respectively equal to three faces including a trihedral of the other, and are similarly placed.*

565. COR. 2. *Two right prisms having equal bases and altitudes are equal.* If the faces are not similarly placed, one of the prisms can be inverted and applied to the other.

Proposition IV. Theorem.

566. *An oblique prism is equivalent to a right prism whose base is equal to a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Let FI be a right section of the oblique prism AD'.

Produce AA' to F', making $FF' = AA'$, and at F' pass the plane $F'I' \perp$ to FF', cutting all the faces of AD' produced, and forming the right section $F'I'$ equal and parallel to FI.

To prove $\qquad AD' \approx FI'$.

Proof. In the solids AI and $A'I'$, $AD = A'D'$. § 551

Also $AG = A'G'$; for, $AF = A'F'$, and $BG = B'G'$, since $AA' = FF'$ and $BB' = GG'$; and AB and FG are equal and parallel to $A'B'$ and $F'G'$ respectively, since AB' and FG' are parallelograms (§ 551). Therefore AG and $A'G'$ are mutually equilateral and equiangular, and hence equal. § 203

In like manner we may prove BH and $B'H'$ equal.

Hence the truncated prisms AI and $A'I'$ are equal. § 564

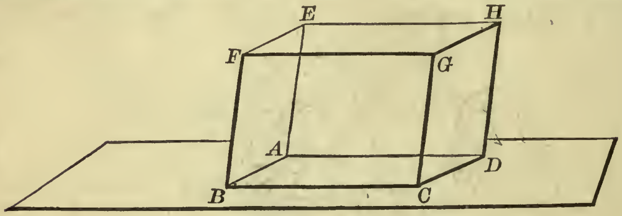
Taking each in turn from the whole solid, we have

$$AD' \approx FI'.$$

Q. E. D.

PROPOSITION V. THEOREM.

567. *Any two opposite faces of a parallelopiped are equal and parallel.*



Let AG be a parallelopiped.

To prove faces AF and DG equal and parallel.

Proof. AB is $\parallel$ to DC and AE is $\parallel$ to DH. §§ 558, 168

Hence $\angle EAB = \angle HDC$. § 498

Also $AB = DC$ and $AE = DH$. § 179

Therefore the face $AF =$ face DG. § 185

Moreover, the face AF is $\parallel$ to DG, § 498

(*if two* $\triangle$ *not in the same plane have their sides* $\parallel$ *and lying in the same direction, their planes are parallel*).

In like manner any two opposite faces may be proved equal and parallel. Q. E. D.

568. SCHOLIUM. Any two opposite faces of a parallelopiped may be taken for bases, since they are equal and parallel parallelograms.

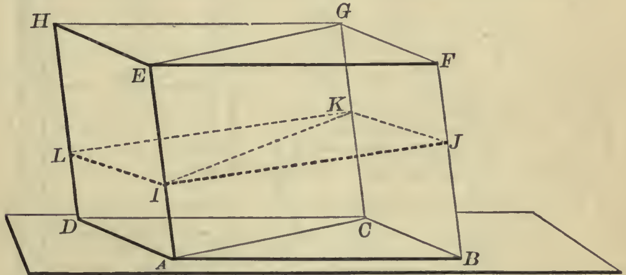
Ex. 490. Show that any lateral edge of a right prism is equal to the altitude.

Ex. 491. Show that the lateral faces of right prisms are rectangles.

Ex. 492. Prove that every section of a prism made by a plane parallel to the lateral edges is a parallelogram.

Proposition VI. Theorem.

569. *The plane passed through two diagonally opposite edges of a parallelopiped divides the parallelopiped into two equivalent triangular prisms.*



Let the plane $AEGC$ pass through the opposite edges AE and CG of the parallelopiped AG.

To prove that the parallelopiped AG is divided into two equivalent triangular prisms ABC-F and ACD-H.

Proof. Let $IJKL$ be a right section of the parallelopiped made by a plane $\perp$ to the edge AE.

Since the opposite faces are parallel, § 567

IJ is $\parallel$ to LK, and IL to JK. § 492

Therefore $IJKL$ is a parallelogram. § 168

The intersection IK of the right section with the plane $AEGC$ is the diagonal of the $\square$ $IJKL$.

$\therefore \triangle IKJ = \triangle IKL.$ § 178

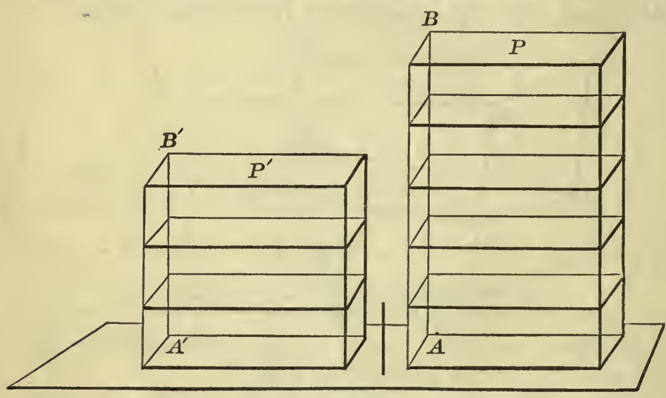
But the prism ABC-F is equivalent to a right prism whose base is IJK and whose altitude is AE, and the prism ACD-H is equivalent to a right prism whose base is ILK, and whose altitude is AE. § 566

But these two right prisms are equal. § 565

$\therefore ABC\text{-}F \approx ACD\text{-}H.$ Q. E. D.

Proposition VII. Theorem.

570. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*



Let AB and $A'B'$ be the altitudes of the two rectangular parallelopipeds P and P', having equal bases.

To prove $P : P' = AB : A'B'$.

Case I. *When AB and $A'B'$ are commensurable.*

Proof. Find a common measure of AB and $A'B'$.

Suppose this common measure to be contained in AB m times, and in $A'B'$ n times; then we have

$$AB : A'B' = m : n. \quad (1)$$

At the several points of division on AB and $A'B'$ pass planes $\perp$ to these lines.

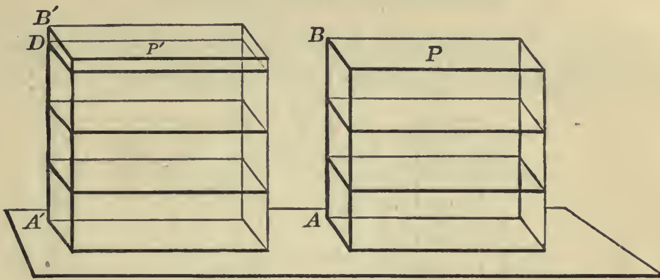
The parallelopiped P will be divided into m,

and P' into n, parallelopipeds, equal each to each. § 565

Therefore $P : P' = m : n$. (2)

From (1) and (2), $P : P' = AB : A'B'$.

CASE II. *When AB and $A'B'$ are incommensurable.*



Let AB be divided into any number of equal parts,

and let one of these parts be applied to $A'B'$ as a unit of measure as many times as $A'B'$ will contain it.

Since AB and $A'B'$ are incommensurable, a certain number of these parts will extend from A' to a point D, leaving a remainder DB' less than one of the parts.

Through D pass a plane $\perp$ to $A'B'$, and denote the parallelopiped whose base is the same as that of P', and whose altitude is $A'D$, by Q.

Now, since AB and $A'D$ are commensurable,

$$Q : P = A'D : AB. \qquad \text{Case I.}$$

If the unit of measure is indefinitely diminished, these ratios continue equal, and they approach indefinitely the limiting ratios $P' : P$ and $A'B' : AB$ respectively.

$$\text{Therefore } P' : P = A'B' : AB, \qquad \S\ 260$$

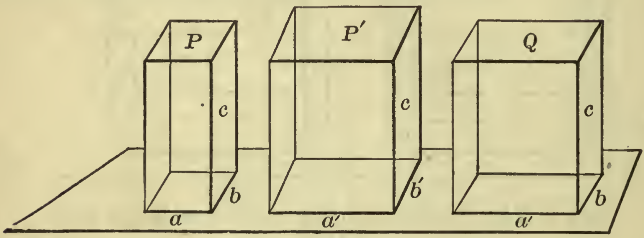
(*if two variables are constantly equal, and each approaches a limit, their limits are equal*).

Q. E. D.

571. SCHOLIUM. The three edges of a rectangular parallelopiped which meet at a common vertex are its *dimensions*. Hence *two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions.*

Proposition VIII. Theorem.

572. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Let a, b, **and** c, **and** a', b', c, **be the three dimensions respectively of the two rectangular parallelopipeds** P **and** P'.

To prove $$\frac{P}{P'}=\frac{a\times b}{a'\times b'}.$$

Let Q be a third rectangular parallelopiped whose dimensions are a', b, and c.

Now Q has the two dimensions b and c in common with P, and the two dimensions a' and c in common with P'.

Then $$\frac{P}{Q}=\frac{a}{a'},$$

and $$\frac{Q}{P'}=\frac{b}{b'},$$ § 571

(*two rectangular parallelopipeds which have two dimensions in common are to each other as their third dimensions*).

The product of these two equalities is

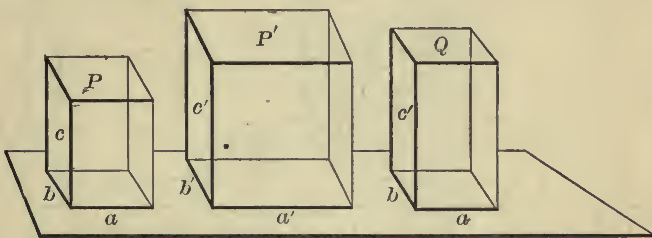
$$\frac{P}{P'}=\frac{a\times b}{a'\times b'}.$$

Q. E. D.

573. Scholium. This proposition may be stated as follows: *Two rectangular parallelopipeds which have one dimension in common are to each other as the products of the other two dimensions.*

Proposition IX. Theorem.

574. *Two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Let a, b, c, **and** a', b', c', **be the three dimensions respectively of the two rectangular parallelopipeds** P **and** P'.

To prove
$$\frac{P}{P'} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

Proof. Let Q be a third rectangular parallelopiped whose dimensions are a, b, and c'.

Then
$$\frac{P}{Q} = \frac{c}{c'}, \qquad \S\ 571$$

and
$$\frac{Q}{P'} = \frac{a \times b}{a' \times b'}, \qquad \S\ 573$$

(*two rectangular parallelopipeds which have one dimension in common are to each other as the products of the other two dimensions*).

The product of these equalities is

$$\frac{P}{P'} = \frac{a \times b \times c}{a' \times b' \times c'}. \qquad \text{Q. E. D.}$$

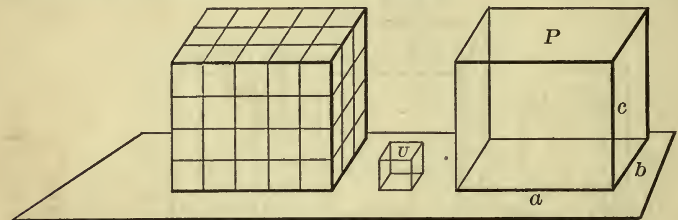
6DP' = 720P
1 : 12 P
P/P = 1/12
8 9 10
60/420

Ex. 493. Find the ratio of two rectangular parallelopipeds if their altitudes are each 6 inches, and their bases 5 inches by 4 inches, and 10 inches by 8 inches, respectively.

Ex. 494. Find the ratio of two rectangular parallelopipeds, if their dimensions are 3, 4, 5, and 9. 8, 10, respectively.

PROPOSITION X. THEOREM.

575. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions, the unit of volume being a cube whose edge is the linear unit.*



Let** a, b, **and** c **be the three dimensions of the rectangular parallelopiped** P, **and let the cube** U **be the unit of volume.

To prove that the volume of $P = a \times b \times c$.

Proof. $\dfrac{P}{U} = \dfrac{a \times b \times c}{1 \times 1 \times 1} = a \times b \times c.$ § 574

Since U is the unit of volume, $\dfrac{P}{U}$ is the volume of P. § 547

Therefore the volume of $P = a \times b \times c$. Q. E. D.

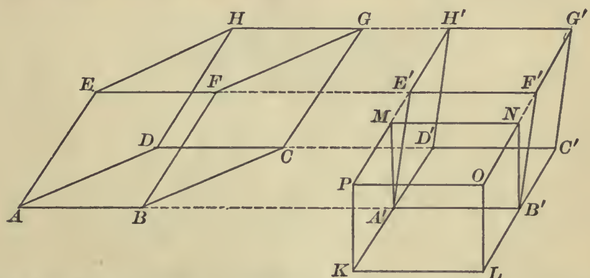
576. COR. 1. *The volume of a cube is the cube of its edge.*

577. COR. 2. The product $a \times b$ represents the area of base when c is the altitude; hence: *The volume of a rectangular parallelopiped is equal to the product of its base by its altitude.*

578. SCHOLIUM. When the three dimensions of the rectangular parallelopiped are each exactly divisible by the linear unit, this proposition is rendered evident by dividing the solid into cubes, each equal to the unit of volume. Thus, if the three edges which meet at a common vertex contain the linear unit 3, 4, and 5 times respectively, planes passed through the several points of division of the edges, and perpendicular to them, will divide the solid into cubes, each equal to the unit of volume; and there will evidently be $3 \times 4 \times 5$ of these cubes.

Proposition XI. Theorem.

579. *The volume of any parallelopiped is equal to the product of its base by its altitude.*



Let H denote the altitude of the parallelopiped AG.

To prove that the volume $AG = ABCD \times H$.

Proof. Consider $ADHE$ the base of AG, and prolong the lateral edges AB, DC, EF, HG.

The right parallelopiped $A'G'$, determined by two right sections $A'D'H'E'$, $B'C'G'F'$, with lateral edge $A'B' = AB$, is equivalent to AG. § 566

Again, consider $D'C'G'H'$ the base of $A'G'$, and prolong the lateral edges $D'A'$, $C'B'$, $H'E'$, $G'F'$.

Then the parallelopiped $A'O$, determined by two right sections $A'B'NM$, $KLOP$, with lateral edge $A'K = D'A'$ is equivalent to $A'G'$ (§ 566), and hence to AG.

The three solids have a common altitude H (§ 494), and equivalent bases, for $ABCD \approx A'B'C'D'$ (§ 366), and $A'B'C'D' = A'KLB'$ (§ 186).

But $A'O$ is a rectangular parallelopiped, for the right sections $A'N$, KO, are rectangles, since the opposite faces $A'P$, $B'O$, are $\perp$ to $A'B'LK$.

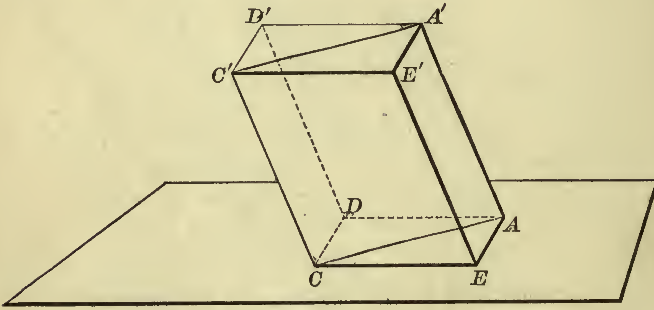
Hence the volume $A'O = A'B'LK \times H$. § 577

Therefore the volume $AG = ABCD \times H$.

Q. E. D.

Proposition XII. Theorem.

580. *The volume of a triangular prism is equal to the product of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the triangular prism AEC-E'.

To prove $V = B \times H.$

Proof. Upon the edges AE, EC, EE', construct the parallelopiped $AECD\text{-}E'$.

Then, since a plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms, § 569

$$AEC\text{-}E' \approx \tfrac{1}{2}\, AECD\text{-}E'.$$

Since the volume of any parallelopiped is equal to the product of its base by its altitude,

$$AECD\text{-}E' = AECD \times H. \quad \text{§ 579}$$

But $$AECD = 2B. \quad \text{§ 178}$$

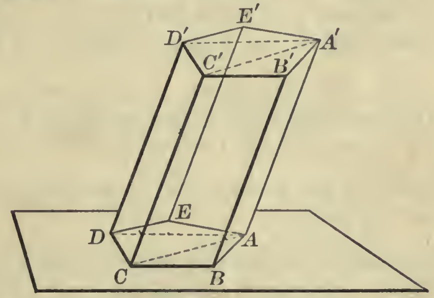
$$\therefore V = \tfrac{1}{2}(2B \times H) = B \times H.$$

Q. E. D.

Ex. 495. Find the volume of a right triangular prism, if its height is 14 inches, and the sides of the base are 6, 5, and 5 inches.

Proposition XIII. Theorem.

581. ***The volume of any prism is equal to the product of its base by its altitude.***



Let V denote the volume, B the base, and H the altitude of the prism DA'.

To prove $V = B \times H$.

Proof. Planes passed through the lateral edge AA', and the diagonals AC, AD of the base, will divide the given prism into triangular prisms.

The volume of each triangular prism is equal to the product of its base by its altitude (§ 580); and hence the sum of the volumes of the triangular prisms is equal to the sum of their bases multiplied by their common altitude.

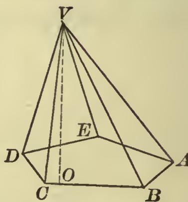
But the sum of the triangular prisms is equal to the given prism, and the sum of their bases is equal to the base of the given prism. Therefore the volume of the given prism is equal to the product of its base by its altitude.

That is, $V = B \times H$. Q. E. D.

582. Cor. *The volumes of two prisms are to each other as the products of their bases and altitudes; prisms having equivalent bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases; prisms having equivalent bases and equal altitudes are equivalent.*

Pyramids.

583. A *pyramid* is a polyhedron of which one face, called the *base*, is a polygon, and the other faces, called *lateral faces*, are triangles having a common vertex, called the *vertex* of the pyramid.



584. The intersections of the lateral faces are called the *lateral edges* of the pyramid.

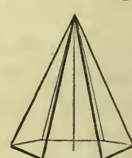
585. The sum of the areas of the lateral faces is called the *lateral area* of the pyramid.

586. The *altitude* of a pyramid is the length of the perpendicular let fall from the vertex to the plane of the base.

587. A pyramid is called *triangular*, *quadrangular*, etc., according as its base is a triangle, quadrilateral, etc.

588. A triangular pyramid, having four faces, is called a *tetrahedron*, and any one of its faces can be taken for its base.

589. A pyramid is *regular* if its base is a regular polygon whose centre coincides with the foot of the perpendicular let fall from the vertex to the base.



Regular Pyramid.

590. The lateral edges of a regular pyramid are equal, since they cut off equal distances from the foot of the perpendicular let fall from the vertex to the base (§ 478). Therefore the lateral faces are equal isosceles triangles.

591. The *slant height* of a regular pyramid is the length of the perpendicular from the vertex to the base of any one of its lateral faces. It is the common altitude of all the lateral faces, and bisects the base of the lateral face in which it is drawn.

592. A *frustum* of a pyramid is the portion of a pyramid

included between its base and a plane parallel to the base and cutting all the lateral edges.

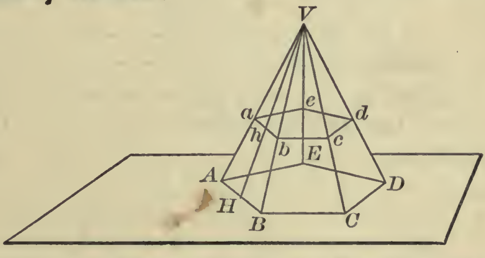
593. The *altitude* of a frustum is the length of the perpendicular between the planes of its bases.

594. The *lateral faces* of a frustum of a regular pyramid are equal trapezoids.

595. The *slant height* of the frustum of a regular pyramid is the altitude of one of these trapezoids.

PROPOSITION XIV. THEOREM.

596. *The lateral area of a regular pyramid is equal to one-half the product of the slant height by the perimeter of its base.*



Let S ***denote the lateral area of the regular pyramid*** $V\text{-}ABCDE$***, and*** VH ***its slant height.***

To prove that $S = \frac{1}{2} VH(AB + BC + \text{etc.})$.

Proof. The $\triangle$ VAB, VBC, etc., are equal isosceles $\triangle$. § 590

The area of the sum of these $\triangle = \frac{1}{2} VH(AB + BC + \text{etc.})$ § 368

But the sum of their areas equals the lateral area of the pyramid.

$$\therefore S = \frac{1}{2} VH(AB + BC + \text{etc.}).$$

Q. E. D.

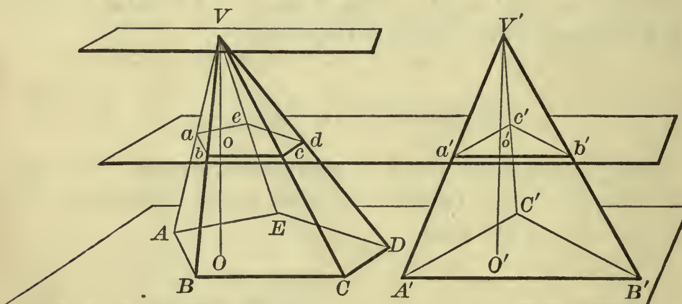
597. COR. 1. *The lateral area of the frustum of a regular pyramid, is equal to one-half the sum of the perimeters of the bases multiplied by the slant height of the frustum.* § 371

Proposition XV. Theorem.

598. *If a pyramid is cut by a plane parallel to its base,*

I. *The edges and altitude are divided proportionally;*

II. *The section is a polygon similar to the base.*



Let $V\text{-}ABCDE$ ***be cut by a plane parallel to its base, intersecting the lateral edges in*** $a, b, c, d, e,$ ***and the altitude in*** o.

To prove I. $\dfrac{Va}{VA} = \dfrac{Vb}{VB} \cdots = \dfrac{Vo}{VO};$

II. *The section abcde similar to the base ABCDE.*

I. **Proof.** Suppose a plane passed through the vertex $V \parallel$ to the base.

Since the edges and the altitude are intersected by three parallel planes,

$$\frac{Va}{VA} = \frac{Vb}{VB} \cdots = \frac{Vo}{VO}. \qquad \S\ 499$$

II. Since the sides ab, bc, etc., are parallel respectively to AB, BC, etc., § 492

the $\angle s$ abc, bcd, etc., are equal respectively to the $\angle s$ ABC, BCD, etc. § 498

Therefore the two polygons are mutually equiangular.

Also, since the sides of the section are parallel to the corresponding sides of the base,

$\triangle$ Vab, Vbc, etc., are similar respectively to $\triangle$ VAB, VBC, etc.

$$\therefore \frac{ab}{AB}=\left(\frac{Vb}{VB}\right)=\frac{bc}{BC}=\left(\frac{Vc}{VC}\right)=\frac{cd}{CD}, \text{ etc.}$$

Hence the polygons have their homologous sides proportional;

Hence section $abcde$ is similar to the base $ABCDE$. § 319

Q. E. D.

599. Cor. 1. *Any section of a pyramid parallel to its base is to the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

Since $\frac{Vo}{VO}=\left(\frac{Vb}{VB}\right)=\frac{ab}{AB}$. $\therefore \frac{\overline{Vo}^2}{\overline{VO}^2}=\frac{\overline{ab}^2}{\overline{AB}^2}$. § 305

But $\frac{abcde}{ABCDE}=\frac{\overline{ab}^2}{\overline{AB}^2}$, § 376

$$\therefore \frac{abcde}{ABCDE}=\frac{\overline{Vo}^2}{\overline{VO}^2}.$$

600. Cor. 2. *If two pyramids having equal altitudes are cut by planes parallel to their bases, and at equal distances from their vertices, the sections will have the same ratio as their bases.*

For $\frac{abcde}{ABCDE}=\frac{\overline{Vo}^2}{\overline{VO}^2}$,

and $\frac{a'b'c'}{A'B'C'}=\frac{\overline{V'o'}^2}{\overline{V'O'}^2}$. § 599

But $Vo=V'o'$, and $VO=V'O'$.

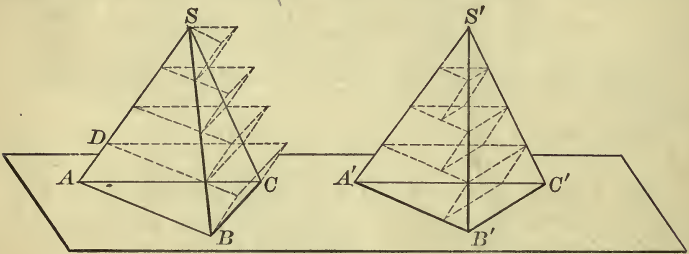
$\therefore abcde : ABCDE = a'b'c' : A'B'C'$.

Whence $abcde : a'b'c' = ABCDE : A'B'C'$. § 298

601. Cor. 3. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to their bases, and at equal distances from their vertices, are equivalent.*

Proposition XVI. Theorem.

602. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Let S-ABC and S'-$A'B'C'$ have equivalent bases situated in the same plane, and a common altitude.

To prove $S\text{-}ABC \approx S'\text{-}A'B'C'$.

Proof. If the pyramids are not equivalent, suppose S-ABC the greater. Divide the common altitude into n equal parts.

Through the points of division pass planes ‖ to the plane of their bases. The corresponding sections of the pyramids are equivalent. § 601

On the base of S-ABC, and on each section, as lower base, construct a prism with lateral edges equal and parallel to AD.

Similarly, construct a prism on each section of S'-$A'B'C'$, as upper base,

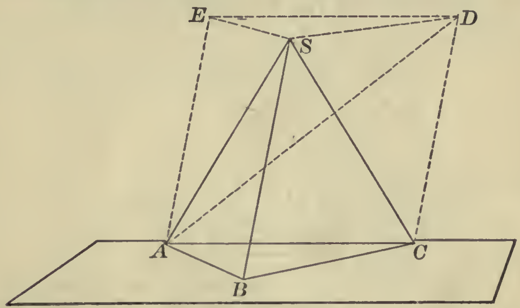
The sum of the first series of prisms is greater than S-ABC, and the sum of the second series is less than S'-$A'B'C'$; therefore the difference between S-ABC and S'-$A'B'C'$ is less than the difference between the sums of these two series of prisms.

Each prism in S'-$A'B'C'$ is equivalent to the prism next above it in S-ABC (§ 582). Hence the difference between the two series of prisms is the lowest prism of the first series. But by increasing n indefinitely this can be made less than any assigned volume, however small.

Therefore the two pyramids cannot differ by any volume however small; therefore the pyramids are equivalent. Q. E. D.

PROPOSITION XVII. THEOREM.

603. *The volume of a triangular pyramid is equal to one-third of the product of its base and altitude.*



Let V denote the volume, and H the altitude, of the triangular pyramid S-ABC.

To prove $V = \frac{1}{3} ABC \times H.$

Proof. On the base ABC construct a prism ABC-SED, having its lateral edges equal and parallel to SB.

The prism will be composed of the triangular pyramid S-ABC and the quadrangular pyramid S-$ACDE$.

Through SA and SD pass a plane SAD.

This plane divides the quadrangular pyramid into the two triangular pyramids S-ACD and S-AED, which have the same altitude and equal bases. § 178

$\therefore S\text{-}ACD \backsimeq S\text{-}AED.$ § 602

Now the pyramid S-AED may be regarded as having ESD for its base and A for its vertex.

$\therefore S\text{-}AED \backsimeq S\text{-}ABC.$

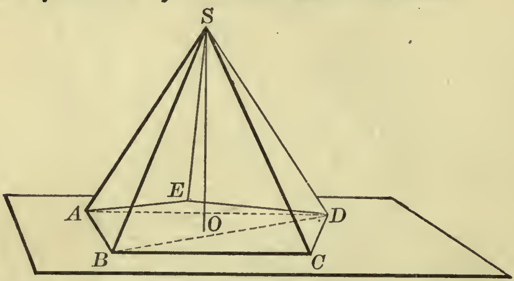
Hence the three pyramids into which the prism ABC-SED is divided are equivalent; the pyramid S-ABC is equivalent to one-third of the prism.

But the volume of the prism is equal to the product of its base and altitude. § 580

$\therefore V = \frac{1}{3} ABC \times H.$ Q. E. D.

Proposition XVIII. Theorem.

604. *The volume of any pyramid is equal to one-third the product of its base and altitude.*



Let V denote the volume of the pyramid $S\text{-}ABCDE$.

To prove $V = \frac{1}{3} ABCDE \times SO$.

Proof. Through the edge SD, and the diagonals of the base DA, DB, pass planes.

These divide the pyramid into triangular pyramids, whose bases are the triangles which compose the base of the pyramid, and whose common altitude is the altitude SO of the pyramid.

The volume of the given pyramid is equal to the sum of the volumes of the triangular pyramids.

But the sum of the volumes of the triangular pyramids is equal to one-third the sum of their bases multiplied by their common altitude. § 603

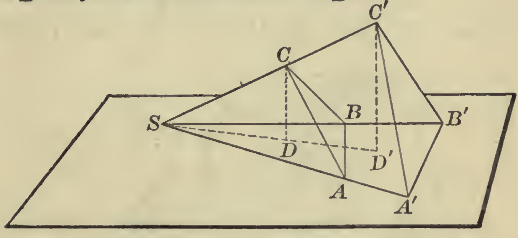
That is, $V = \frac{1}{3} ABCDE \times SO$. Q. E. D.

605. Cor. *The volumes of two pyramids are to each other as the products of their bases and altitudes; pyramids having equivalent bases are to each other as their altitudes; pyramids having equal altitudes are to each other as their bases; pyramids having equivalent bases and equal altitudes are equivalent.*

606. Scholium. The volume of any polyhedron may be found by dividing it into pyramids, computing their volumes separately, and finding the sum of their volumes.

Proposition XIX. Theorem.

607. *The volumes of two tetrahedrons, having a trihedral angle of the one equal to a trihedral angle of the other, are to each other as the products of the three edges of these trihedral angles.*



***Let** V **and** V' **denote the volumes of the two tetrahedrons** $S\text{-}ABC$ **and** $S\text{-}A'B'C'$, **having the common trihedral angle** S.*

To prove $$\frac{V}{V'} = \frac{SA \times SB \times SC}{SA' \times SB' \times SC'}.$$

Proof. Draw CD and $C'D' \perp$ to the plane $SA'B'$, and let their plane intersect $SA'B'$ in SDD'.

The faces SAB and $SA'B'$ may be taken as the bases, and CD, $C'D'$ as the altitudes, of the triangular pyramids $SAB\text{-}C$ and $SA'B'\text{-}C'$.

$$\therefore \frac{V}{V'} = \frac{SAB \times CD}{SA'B' \times C'D'} = \frac{SAB}{SA'B'} \times \frac{CD}{C'D'}, \qquad \text{§ 605}$$

(*any two pyramids are to each other as the products of their bases and altitudes*).

But $$\frac{SAB}{SA'B'} = \frac{SA \times SB}{SA' \times SB'}, \qquad \text{§ 374}$$

and $$\frac{CD}{C'D'} = \frac{SC}{SC'}, \qquad \text{§ 319}$$

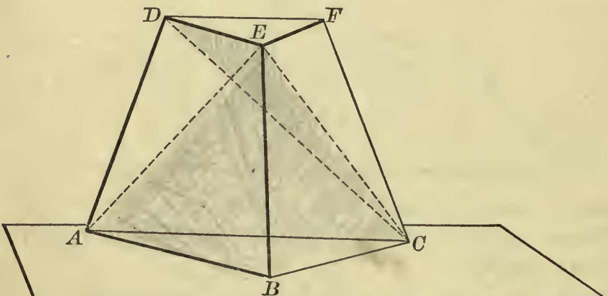
(*being homologous sides of the similar* $\triangle$ *SDC and SD'C'*).

$$\therefore \frac{V}{V'} = \frac{SA \times SB \times SC}{SA' \times SB' \times SC'}.$$

Q. E. D.

PROPOSITION XX. THEOREM.

608. *The frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the two bases of the frustum.*



Let *B* and *b* denote the lower and upper bases of the frustum *ABC-DEF*, and *H* its altitude.

Through the vertices A, E, C and E, D, C pass planes dividing the frustum into three pyramids.

Now the pyramid $E\text{-}ABC$ has for its altitude H, the altitude of the frustum, and for its base B, the lower base of the frustum.

And the pyramid $C\text{-}EDF$ has for its altitude H, the altitude of the frustum, and for its base b, the upper base of the frustum. Hence it only remains

To prove $E\text{-}ADC$ equivalent to a pyramid, having for its altitude H, and for its base $\sqrt{B \times b}$.

Proof. $E\text{-}ABC$ and $E\text{-}ADC$, regarded as having the common vertex C, and their bases in the same plane BD, have a common altitude.

$$\therefore C\text{-}ABE : C\text{-}ADE = \triangle AEB : \triangle AED, \qquad \S\ 605$$

(*pyramids having equal altitudes are to each other as their bases*).

Now since the $\triangle$ AEB and AED have a common altitude,
(*that is, the altitude of the trapezoid ABED*),

we have $\triangle AEB : \triangle AED = AB : DE,$ § 370

$$\therefore C\text{-}ABE : C\text{-}ADE = AB : DE.$$

That is, $E\text{-}ABC : E\text{-}ADC = AB : DE.$

In like manner $E\text{-}ADC$ and $E\text{-}DFC$, regarded as having the common vertex E, and their bases in the same plane DC, have a common altitude.

$$\therefore E\text{-}ADC : E\text{-}DFC = \triangle ADC : \triangle DFC. \qquad \S\ 605$$

But since the $\triangle$ ADC and DFC have a common altitude,
(*that is, the altitude of the trapezoid ACFD*),

we have $\triangle ADC : \triangle DFC = AC : DF.$ § 370

$$\therefore E\text{-}ADC : E\text{-}DFC = AC : DF.$$

But $\triangle DEF$ is similar to $\triangle ABC$, § 598

(*the section of a pyramid made by a plane* || *to the base is a polygon similar to the base*).

$$\therefore AB : DE = AC : DF. \qquad \S\ 319$$

$$\therefore E\text{-}ABC : E\text{-}ADC = E\text{-}ADC : E\text{-}DFC.$$

Now $E\text{-}ABC = \frac{1}{3} H \times B,$ § 603

and $E\text{-}DFC = C\text{-}EDF = \frac{1}{3} H \times b.$

$$\therefore E\text{-}ADC = \sqrt{\tfrac{1}{3} H \times B \times \tfrac{1}{3} H \times b} = \tfrac{1}{3} H \sqrt{B \times b}.$$

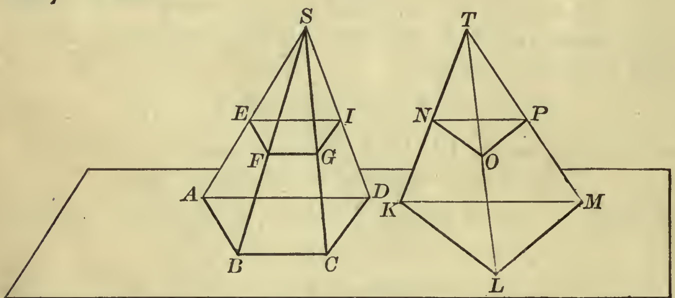
Hence, $E\text{-}ADC$ is equivalent to a pyramid, having for its altitude H, and for its base $\sqrt{B \times b}$. Q. E. D.

609. Cor. *If the volume of the frustum of a triangular pyramid is denoted by V, the lower base by B, the upper base by b, and the altitude by H,*

$$V = \tfrac{1}{3} H \times B + \tfrac{1}{3} H \times b + \tfrac{1}{3} H \times \sqrt{B \times b}$$
$$= \tfrac{1}{3} H \times (B + b + \sqrt{B \times b}).$$

Proposition XXI. Theorem.

610. *The volume of the frustum of any pyramid is equal to the sum of the volumes of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*



Let B and b denote the lower and upper bases, H the altitude, and V the volume of $ABCD$-$EFGI$.

To prove $V = \frac{1}{3} H (B + b + \sqrt{B \times b})$.

Proof. Let T-KLM be a triangular pyramid having the same altitude as S-$ABCD$ and its base $KLM \Leftrightarrow ABCD$, and lying in the same plane. Then T-$KLM \Leftrightarrow S$-$ABCD$. § 605

Let the plane $EFGI$ cut T-KLM in NOP.

Then $NOP \Leftrightarrow EFGI$. § 601

Hence T-$NOP \Leftrightarrow S$-$EFGI$.

Taking away the upper pyramids leaves the frustums equivalent.

But the volume of the frustum of the triangular pyramid is equal to $\frac{1}{3} H (B + b + \sqrt{B \times b})$. § 609

$$\therefore V = \frac{1}{3} H (B + b + \sqrt{B \times b}).$$ Q. E. D.

1 cu. ft. of H_2O weighs 62.5 lbs.

V = 42 cu ft

NUMERICAL EXERCISES.

496. Find the length of an edge of a cubical vessel which will hold 2 tons of water.

497. How many square feet of lead will be required to line a cistern, open at the top, which is 4 feet 6 inches long, 2 feet 8 inches wide, and contains 42 cubic feet?

498. An open cistern is made of iron 2 inches thick. The inner dimensions are: length, 4 feet 6 inches; breadth, 3 feet; depth, 2 feet 6 inches. What will the cistern weigh (i.) when empty? (ii.) when full of water? Specific gravity of iron = 7.2. inside Area = 51

499. An open cistern 6 feet long and $4\frac{1}{2}$ feet wide holds 108 cubic feet of water. How many cubic feet of lead will it take to line the sides and bottom, if the lead is $\frac{1}{8}$ inch thick?

500. The three dimensions of a rectangular parallelopiped are a, b, c; find the surface, the volume, and the length of a diagonal.

501. The base of a right prism is a rhombus, one side of which is 10 inches, and the shorter diagonal is 12 inches. The height of the prism is 15 inches. Find the entire surface and the volume.

502. Find the volume of a regular hexagonal prism whose height is 10 feet, each side of the hexagon being 10 inches.

503. A pyramid 15 feet high has a base containing 169 square feet. At what distance from the vertex must a plane be passed parallel to the base so that the section may contain 100 square feet?

504. The base of a pyramid contains 144 square feet. A plane parallel to the base and 4 feet from the vertex cuts a section containing 64 square feet; find the height of the pyramid.

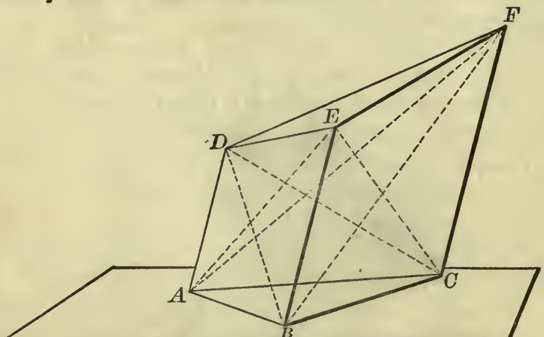
505. A pyramid 12 feet high has a square base measuring 8 feet on a side. What will be the area of a section made by a plane parallel to the base and 4 feet from the vertex?

506. Two pyramids standing on the same plane are 14 feet high. The first has for base a square measuring 9 feet on a side; the second a regular hexagon measuring 7 feet on a side. Find the areas of the sections made by a plane parallel to their bases and 6 feet from their vertices.

507. The base of a regular pyramid is a hexagon of which the side measures 3 feet. Find the height of the pyramid if the lateral area is equal to ten times the area of the base.

PROPOSITION XXII. THEOREM.

611. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism, and whose vertices are the three vertices of the inclined section.*



Let ABC-DEF **be a truncated triangular prism whose base is** ABC, **and inclined section** DEF.

Pass the planes AEC and DEC, dividing the truncated prism into the three pyramids E-ABC, E-ACD, and E-CDF.

To prove ABC-DEF *equivalent to the sum of the three pyramids,* E-ABC, D-ABC, *and* F-ABC.

Proof. E-ABC has the base ABC and the vertex E.

The pyramid E-$ACD \approx B$-ACD, § 602

(*for they have the same base* ACD *and the same altitude, since their vertices* E *and* B *are in the line* $EB \parallel$ *to the base* ACD).

But the pyramid B-ACD may be regarded as having the base ABC and the vertex D; that is, as D-ABC.

The pyramid E-$CDF \approx B$-ACF,

for their bases CDF and ACF, in the same plane, are equivalent, § 369

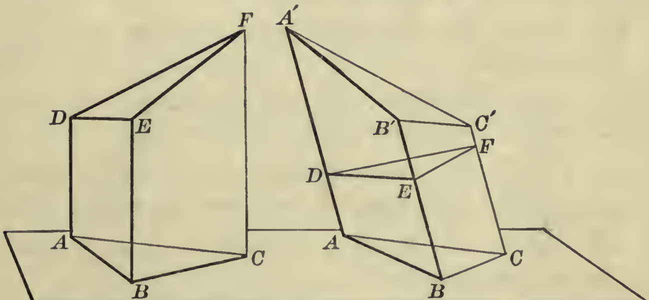
(*since the* $\triangle$ CDF *and* ACF *have the common base* CF *and equal altitudes, their vertices lying in the line* $AD \parallel$ *to* CF),

and the pyramids have the same altitude,

(*since their vertices E and B are in the line $EB \parallel$ to the plane of their bases $ACDF$*).

But the pyramid B-ACF may be regarded as having the base ABC and the vertex F; that is, as F-ABC.

Therefore the truncated triangular prism ABC-DEF is equivalent to the sum of the three pyramids E-ABC, D-ABC, and F-ABC. Q. E. D



612. COR. 1. *The volume of a truncated right triangular prism is equal to the product of its base by one-third the sum of its lateral edges.* For the lateral edges DA, EB, FC, being perpendicular to the base, are the altitudes of the three pyramids whose sum is equivalent to the truncated prism. And, since the volume of a pyramid is one-third the product of its base by its altitude, the sum of the volumes of these pyramids $= ABC \times \frac{1}{3}(DA + EB + FC)$.

613. COR. 2. *The volume of any truncated triangular prism is equal to the product of its right section by one-third the sum of its lateral edges.* For let ABC-$A'B'C'$ be any truncated triangular prism. Then the right section DEF divides it into two truncated right prisms whose volumes are

$$DEF \times \tfrac{1}{3}(AD + BE + CF) \text{ and } DEF \times \tfrac{1}{3}(A'D + B'E + C'F).$$

Whence their sum is $DEF \times \frac{1}{3}(AA' + BB' + CC')$.

SIMILAR POLYHEDRONS.

614. *Similar polyhedrons* are polyhedrons that have the same number of faces, respectively similar and similarly placed, and their corresponding polyhedral angles equal.

Homologous faces, lines, and *angles* of similar polyhedrons are faces, lines, and angles similarly placed.

615. COR. 1. *The homologous edges of similar polyhedrons are proportional.* § 319

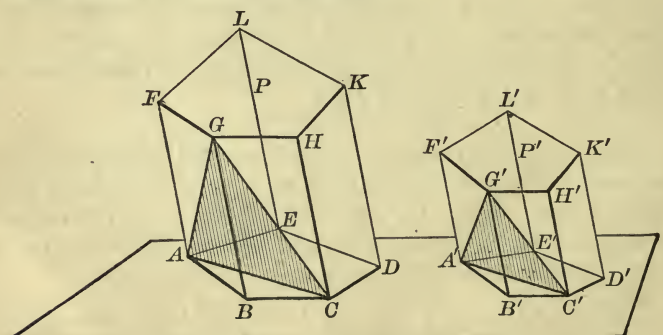
616. COR. 2. *Two homologous faces of similar polyhedrons are proportional to the squares of two homologous edges.* § 377

617. COR. 3. *The entire surfaces of two similar polyhedrons are proportional to the squares of two homologous edges.* § 303

618. COR. 4. *The homologous dihedral angles of similar polyhedrons are equal.*

PROPOSITION XXIII. THEOREM.

619. ***Two similar polyhedrons may be decomposed into the same number of tetrahedrons similar, each to each, and similarly placed.***



Let P and P' be two similar polyhedrons.

To prove that the similar polyhedrons P and P′ can be decomposed into the same number of tetrahedrons, similar each to each, and similarly placed.

Proof. Through the vertices A, G, C, and the homologous vertices A', G', C', pass planes.

The tetrahedrons $G\text{-}ABC$ and $G'\text{-}A'B'C'$ have the faces ABC, GAB, GBC, similar respectively to $A'B'C'$, $G'A'B'$, $G'B'C'$. § 332

Hence in the faces GAC and $G'A'C'$

$$\frac{AG}{A'G'}=\left(\frac{AB}{A'B'}\right)=\frac{AC}{A'C'}=\left(\frac{BC}{B'C'}\right)=\frac{GC}{G'C'}. \qquad \S\ 319$$

Therefore the face GAC is similar to $G'A'C'$. § 324

Hence the faces of these tetrahedrons are similar, each to each.

Also, any two corresponding trihedral $\measuredangle$ of these tetrahedrons are equal. § 541

Therefore the tetrahedron $G\text{-}ABC$ is similar to $G'\text{-}A'B'C'$. § 614

If $G\text{-}ABC$ and $G'\text{-}A'B'C'$ be removed, the polyhedrons remaining will continue similar; for the new faces GAC and $G'A'C'$ have just been proved similar, and the modified faces AGF and $A'G'F'$, CGH and $C'G'H'$, will be similar (§ 332); also the modified polyhedral $\measuredangle$ G and G', A and A', C and C', will remain equal each to each, since the corresponding parts taken from them are equal.

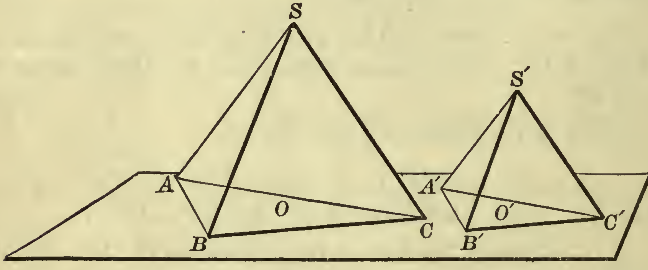
The process of removing similar tetrahedrons can be carried on until the polyhedrons are reduced to tetrahedrons; that is, until the two similar polyhedrons are decomposed into the same number of tetrahedrons similar each to each, and similarly situated.

Q. E. D.

620. Cor. *Any two homologous lines in two similar polyhedrons have the same ratio as any two homologous edges.*

Proposition XXIV. Theorem.

621. *The volumes of two similar tetrahedrons are to each other as the cubes of their homologous edges.*



Let V and V' denote the volumes of the two similar tetrahedrons S-ABC and S'-$A'B'C'$.

To prove $$\frac{V}{V'}=\frac{\overline{SB}^3}{\overline{S'B'}^3}.$$

Proof. Since the homologous trihedral angles S and S' are equal, we have

$$\frac{V}{V'}=\frac{SB\times SC\times SA}{S'B'\times S'C'\times S'A'} \qquad \S\ 607$$

$$=\frac{SB}{S'B'}\times\frac{SC}{S'C'}\times\frac{SA}{S'A'}.$$

But $$\frac{SB}{S'B'}=\frac{SC}{S'C'}=\frac{SA}{S'A'}. \qquad \S\ 615$$

$$\therefore\ \frac{V}{V'}=\frac{SB}{S'B'}\times\frac{SB}{S'B'}\times\frac{SB}{S'B'}=\frac{\overline{SB}^3}{\overline{S'B'}^3}.$$

Q. E. D.

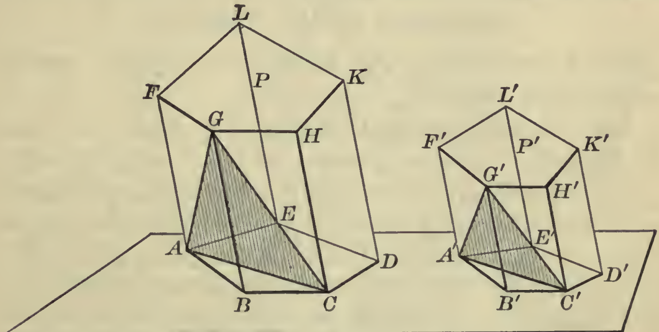
Ex. 508. The homologous edges of two similar tetrahedrons are as 6 : 7. Find the ratio of their surfaces and of their volumes.

Ex. 509. If the edge of a tetrahedron is a, find the homologous edge of a similar tetrahedron twice as large. $a : 2a$

$a^3 = 8a^3 : a : 8a$

PROPOSITION XXV. THEOREM.

622. *The volumes of two similar polyhedrons are to each other as the cubes of any two homologous edges.*



Let V, V' denote the volumes, GB, G'B' any two homologous edges, of the polyhedrons P and P'.

To prove $V : V' = \overline{GB}^3 : \overline{G'B'}^3.$

Proof. Decompose these polyhedrons into tetrahedrons similar, each to each, and similarly placed. § 619

Denote the volumes of these tetrahedrons by $v, v_1, v_2, \ldots\ldots, v', v_1', v_2', \ldots\ldots$ Then

$$\frac{v}{v'} = \frac{\overline{GB}^3}{\overline{G'B'}^3}, \quad \frac{v_1}{v_1'} = \frac{\overline{GB}^3}{\overline{G'B'}^3}, \quad \frac{v_2}{v_2'} = \frac{\overline{GB}^3}{\overline{G'B'}^3}. \qquad \S\ 621$$

$$\therefore \frac{v}{v'} = \frac{v_1}{v_1'} = \frac{v_2}{v_2'}.$$

Whence
$$\frac{v + v_1 + v_2}{v' + v_1' + v_2'} = \frac{v}{v'} = \frac{\overline{GB}^3}{\overline{G'B'}^3}, \qquad \S\ 303$$

or
$$\frac{V}{V'} = \frac{\overline{GB}^3}{\overline{G'B'}^3}.$$

Q. E. D

Regular Polyhedrons.

623. A *regular polyhedron* is a polyhedron whose faces are equal regular polygons, and whose polyhedral angles are equal.

Proposition XXVI. Problem.

624. *To determine the number of regular convex polyhedrons possible.*

A convex polyhedral angle must have at least three faces, and the sum of its face angles must be less than 360° (§ 540).

1. Since each angle of an equilateral triangle is 60°, convex polyhedral angles may be formed by combining three, four, or five equilateral triangles. The sum of six such angles is 360°, and therefore greater than the sum of the face angles of a convex polyhedral angle. Hence not more than three regular convex polyhedrons are possible with equilateral triangles for faces.

2. Since each angle of a square is 90°, a convex polyhedral angle may be formed by combining three squares. The sum of four such angles is 360°, and therefore greater than the sum of the face angles of a convex polyhedral angle. Hence only one regular convex polyhedron is possible with squares.

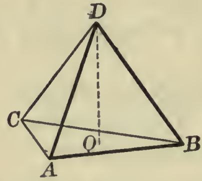
3. Since each angle of a regular pentagon is 108°, a convex polyhedral angle may be formed by combining three regular pentagons. The sum of four such angles is 432°, and therefore greater than the sum of the face angles of a convex polyhedral angle. Hence only one regular convex polyhedron is possible with regular pentagons.

4. We can proceed no further; for the sum of three angles of regular hexagons is 360°, of regular heptagons is greater than 360°, etc. Hence only five regular convex polyhedrons are possible.

There are five regular polyhedrons called, from the number of faces, the *tetrahedron*, the *hexahedron*, the *octahedron*, the *dodecahedron*, the *icosahedron*.

PROPOSITION XXVII. PROBLEM.

625. *Upon a given edge to construct the regular polyhedrons.*



Let *AB* **be the given edge.**

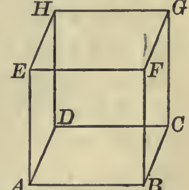
Upon AB to construct the regular polyhedrons.

1. **Construction of the Regular Tetrahedron.** Upon the given edge construct an equilateral triangle. At its centre erect a ⊥ to its plane, and take a point *D* in this ⊥ such that *DA* = *AB*. Join *D* to each of the vertices of the triangle *ABC*. The polyhedron *D-ABCD* is a regular tetrahedron.

Proof. The four faces are by construction equal equilateral triangles (§ 480), and the four trihedral angles *A*, *B*, *C*, *D*, are equal, since their face angles are all equal. § 541

Therefore *D-ABC* is a regular tetrahedron.

2. **Construction of the Regular Hexahedron.** Upon the given edge *AB* construct the square *ABCD*, and upon the sides of this square construct the squares *AF*, *BG*, *CH*, *DE*, ⊥ to the plane *ABCD*.

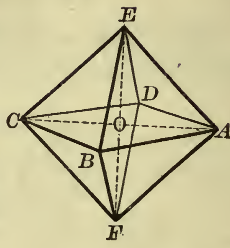


The polyhedron *AG* is a regular hexahedron.

Proof. The six faces are by construction equal squares, and the eight trihedral angles *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, are equal since their face angles are all equal. § 541

Therefore *AG* is a regular hexahedron.

3. Construction of the Regular Octahedron. Upon the given edge AB construct the square $ABCD$, and through its centre O pass a $\perp$ to its plane.



On this $\perp$ take the points E and F such that AE and AF are each equal to AB.

Join E and F to each of the vertices of the square $ABCD$. The polyhedron $E\text{-}ABCD\text{-}F$ is a regular octahedron.

Proof. Since all the lines from E and F to A, B, C, and D are equal (§ 480), and each equal to AB, the eight triangles which form the faces are equal and equilateral.

Since O is the centre of the square $ABCD$, the diagonal of this square AC will pass through O, and the lines EF and AC which intersect in O are in the same plane. Hence E, C, F, and A are in one plane.

In the $\triangle$ AEC, ABC, AFC, the side AC is common, and all the other sides equal. Therefore these triangles are equal (§ 160); and since $\angle ABC$ is a right angle, AEC and AFC are right angles.

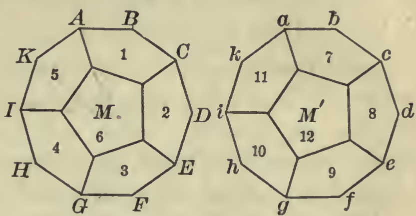
Therefore $AECF$ is a square equal to the square $ABCD$. Hence the pyramid $B\text{-}AECF$ has its four faces and its base $AECF$ equal to the four faces and the base $ABCD$ of the pyramid $E\text{-}ABCD$.

Therefore the two pyramids are equal, and the tetrahedral angle B is equal to the tetrahedral angle E.

In like manner it can be shown that any other two polyhedral angles are equal. Therefore the polyhedron is a regular octahedron.

4. Construction of the Regular Dodecahedron. Construct a regular pentagon M with its sides equal each to the given edge, and join to each of its sides the side of an equal pentagon so inclined to the plane of M as to form trihedral angles at its

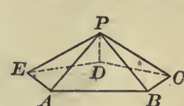
vertices. Construct a regular pentagon $M' = M$, and join to each of its sides the side of an equal pentagon so inclined to the plane of M' as to form trihedral angles at its vertices.



We now have two equal convex surfaces composed each of six equal regular pentagons. The trihedral angles formed at the vertices of M and M' are equal, each to each (§ 541); therefore the dihedral angles are all equal, and the two surfaces can be combined so as to form a single convex surface.

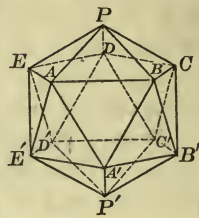
Proof. Put the two surfaces together with their convexities turned in opposite directions, so that the vertex a and the side ab shall coincide with the vertex B and the side BA respectively. Then two consecutive face angles of one surface will unite with a single face angle of the other, and form a trihedral angle, since any two consecutive faces contain a dihedral angle of one of the trihedral angles already formed at the vertices of M and M'. The trihedral angles, therefore, are all equal, and the polyhedron is a regular dodecahedron.

5. **Construction of the Regular Icosahedron.** Construct a regular pentagon $ABCDE$, with its sides equal each to the given edge. At its centre erect a $\perp$ to its plane, and in this perpendicular take a point such that $PA = AB$. Join P with each of the vertices of the pentagon, forming a regular pentagonal pyramid, whose vertex is P, and whose dihedral angles formed on the edges PA, PB, etc., are all equal. § 542



Complete the pentahedral angles at A, B, C, etc., adding to each three equilateral triangles each equal to PAB, and making the dihedral angles about A, B, C, etc., all equal.

Construct a regular pentagonal pyramid $P'\text{-}A'B'C'D'E'$ equal to $P\text{-}ABCDE$. This can be joined to the convex surface already formed, so as to form a single convex surface.

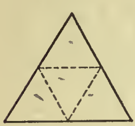


Proof. Two consecutive face angles of one surface will unite with three consecutive face angles of the other, and form a regular pentahedral angle, since they have together three dihedral angles of such a pentahedral angle.

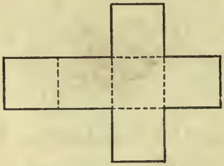
The pentahedral angles are therefore all equal, and the polyhedron is a regular icosahedron.

626. Scholium. The regular polyhearons may be constructed as follows:

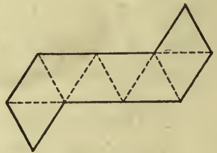
Draw the diagrams given below on cardboard. Cut through the full lines and half through the dotted lines. Bring the edges together so as to form the respective polyhedrons, and keep the edges in contact by pasting along them strips of strong paper.



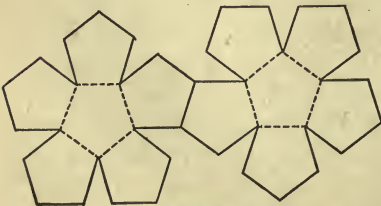
Tetrahedron.



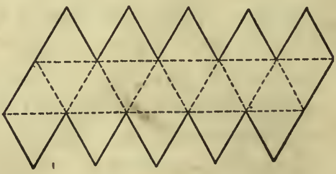
Hexahedron.



Octahedron.



Dodecahedron.

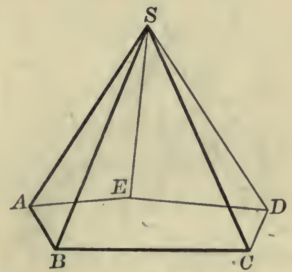


Icosahedron.

General Theorems of Polyhedrons.

Proposition XXVIII. Theorem. (Euler's.)

627. *In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.*



Let E denote the number of edges, V the number of vertices, F the number of faces, of S-$ABCDE$.

To prove $E+2=V+F.$

Proof. Beginning with one face $ABCDE$, we have $E=V.$

Annex a second face SAB, by applying one of its edges to a corresponding edge of the first face, and there is formed a surface having *one* edge AB and *two* vertices A and B *common* to the two faces.

Therefore, for two faces $E=V+1.$

Annex a third face SBC, adjoining each of the first two faces; this face will have *two* edges, SB, BC, and *three* vertices S, B, C, *in common* with the surface already formed.

Therefore, for three faces $E=V+2.$

In like manner, for four faces $E=V+3.$

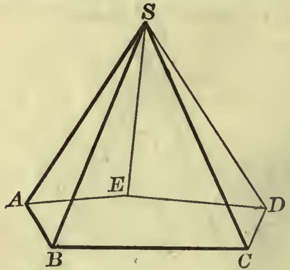
And so on for $(F-1)$ faces $E=V+(F-2).$

But $F-1$ is the number of faces of the polyhedron when only one face is lacking, and the addition of this face will not increase the number of edges or vertices. Hence, for F faces

$$E=V+F-2, \text{ or } E+2=V+F.$$ Q. E. D.

Proposition XXIX. Theorem.

628. *The sum of the face angles of any polyhedron is equal to four right angles taken as many times, less two, as the polyhedron has vertices.*



Let E denote the number of edges, V the number of vertices, F the number of faces, and S the sum of the face angles, of the polyhedron S-ABCDE.

To prove $S = (V - 2)\ 4$ *rt.* $\angle$s.

Proof. Since E denotes the number of edges, $2E$ will denote the number of sides of the faces, considered as independent polygons, for each edge is common to two polygons.

If an exterior angle is formed at each vertex of every polygon, the sum of the interior and exterior angles at each vertex is 2 rt. $\angle$s; and since there are $2E$ vertices, the sum of the interior and exterior angles of all the faces is

$$2E \times 2 \text{ rt. } \angle\text{s, or } E \times 4 \text{ rt. } \angle\text{s.}$$

But the sum of the ext. $\angle$s of each face is 4 rt. $\angle$s (§ 207), and the number of faces is F; therefore the sum of all the ext. $\angle$s is

$$F \times 4 \text{ rt. } \angle\text{s.}$$

Therefore S, the sum of the int. $\angle$s, is

$$(E - F)\ 4 \text{ rt. } \angle\text{s.}$$

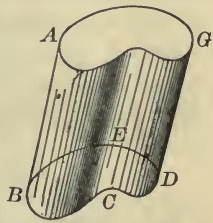
But $E + 2 = V + F$ (§ 627); that is, $E - F = V - 2$.

Therefore $S = (V - 2)\ 4$ rt. $\angle$s.

Q. E. D

THE CYLINDER.

629. A *cylindrical surface* is a curved surface generated by a moving straight line AB, called the *generatrix*, which moves parallel to itself and constantly touches a fixed curve $BCDE$, called the *directrix*. The generatrix in any position is called an *element* of the surface. One element, and only one, can be drawn through a given point of a cylindrical surface.



630. A *cylinder* is a solid bounded by a cylindrical surface and two parallel planes which cut all the elements. The two plane surfaces are called the *bases*, and the cylindrical surface is called the *lateral surface.*

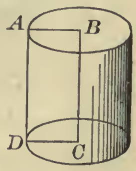
631. The *altitude* of a cylinder is the length of the perpendicular between the planes of its bases. The elements of a cylinder are all equal.

632. A *right section* of a cylinder is a section made by a plane perpendicular to its elements.

633. A cylinder is a *right* cylinder if its elements are perpendicular to its bases; otherwise it is an *oblique* cylinder.

634. A *circular cylinder* is a cylinder whose base is a circle.

635. A *cylinder of revolution* is a cylinder generated by the revolution of a rectangle about one side as an axis.



636. *Similar* cylinders of revolution are cylinders generated by similar rectangles revolving about homologous sides.

637. A *tangent line* to a cylinder is a straight line, not an element, which touches the surface of the cylinder but does not intersect it.

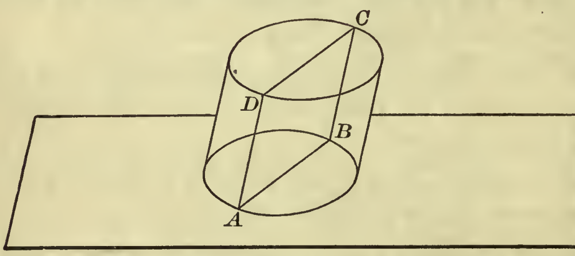
638. A plane which contains an element of the cylinder and does not cut the surface, is called a *tangent plane.* The element contained by the plane is called the *element of contact.*

639. A prism is *inscribed* in a cylinder when its lateral edges are elements of the cylinder and its bases are inscribed in the bases of the cylinder.

640. A prism is *circumscribed* about a cylinder when its lateral edges are parallel to elements of the cylinder and its bases are circumscribed about the bases of the cylinder.

Proposition XXX. Theorem.

641. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*



Let a plane pass through the element AD of the cylinder AC.

To prove the section ABCD a parallelogram.

Proof. A plane passing through the element AD will cut the circumference of the base in a second point B.

The straight line BC drawn ∥ to AD lies in the plane DAB (§ 98); and it is an element of the cylinder. § 629

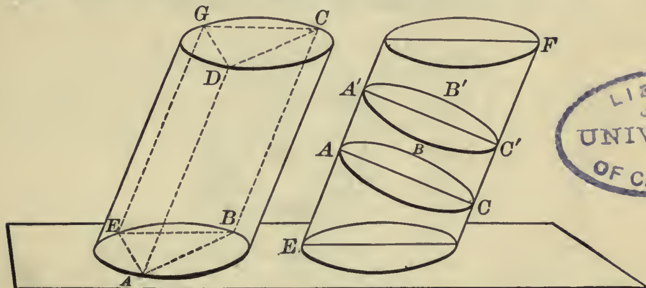
Hence BC is the intersection of the plane and the surface of the cylinder. Also DC is ∥ to AB. § 492

Therefore $ABCD$ is a parallelogram. § 168

642. Cor. *Every section of a right cylinder made by a plane passing through an element is a rectangle.*

PROPOSITION XXXI. THEOREM.

643. *The bases of a cylinder are equal.*



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Let ABE and DCG be the bases of the cylinder AC.

To prove $ABE = DCG$.

Proof. Let A, B, E, be any three points in the perimeter of the lower base, and AD, BC, EG, be elements of the surface.

Join AE, AB, EB, DG, DC, GC.

Then AC, AG, EC are ▱s. § 182

$\therefore AE = DG$, $AB = DC$, and $EB = GC$. § 179

$\therefore \triangle ABE = \triangle DCG$. § 160

Apply the upper base to the lower base so that DC, shall fall upon AB.

Then G will fall upon E.

But G is any point in the perimeter of the upper base, therefore every point in the perimeter of the upper base will fall upon the perimeter of the lower base.

Therefore the bases coincide and are equal.

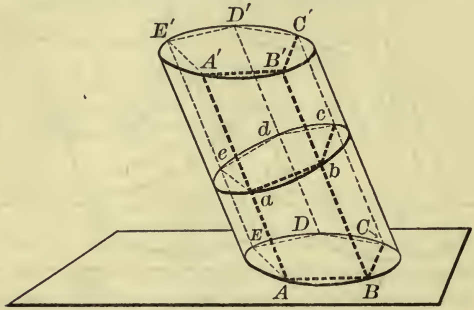
Q. E. D.

644. COR. 1. *Any two parallel sections ABC and $A'B'C'$, cutting all the elements of a cylinder EF, are equal.* For these sections are the bases of the cylinder AC'.

645. COR. 2. *Any section of a cylinder parallel to the base is equal to the base.*

PROPOSITION XXXII. THEOREM.

646. *The lateral area of a cylinder is equal to the product of the perimeter of a right section of the cylinder by an element of the surface.*



Let S denote the lateral area, P the perimeter of a right section, and E an element of the surface of AC'.

To prove $S = P \times E$.

Proof. Inscribe in the cylinder a prism having for its base the polygon $ABCDE$, and denote the lateral area of this prism by s, and the perimeter of the right section $abcde$ by p.

Then $s = p \times E$. § 561

Let the number of lateral faces of the inscribed prism be indefinitely increased, the new edges continually dividing the arcs in the bases of the cylinder. Then the perimeters of the bases of the prism will approach the perimeters of the bases of the cylinder as limits, and the lateral area of the prism will approach the lateral area of the cylinder as a limit. Hence the perimeter of the right section of the prism will approach the perimeter of the right section of the cylinder as a limit.

But, however great the number of faces, $s = p \times E$.

$\therefore S = P \times E$. § 260

Q. E. D.

647. COR. 1. *The lateral area of a cylinder of revolution is the product of the circumference of its base by its altitude.*

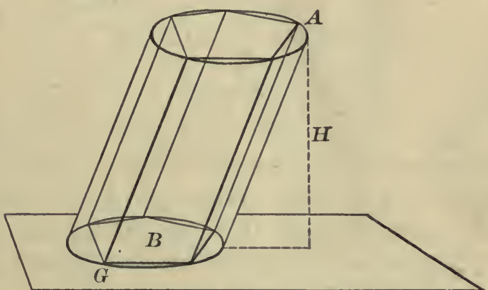
648. Cor. 2. *If S denotes the lateral area, T the total area, H the altitude, and R the radius, of a cylinder of revolution,*

$$S = 2\pi R \times H.$$

$$T = 2\pi R \times H + 2\pi R^2 = 2\pi R(H + R).$$

Proposition XXXIII. Theorem.

649. ***The volume of a cylinder is equal to the product of its base by its altitude.***



Let V denote the volume, B the base, and H the altitude, of the cylinder AG.

To prove $V = B \times H.$

Proof. Let V' denote the volume of the inscribed prism AG, and B' its base. The altitude of this prism will be H.

Then $V' = B' \times H.$ § 581

If the number of lateral faces of the inscribed prism is indefinitely increased, the new edges continually dividing the arcs of the bases, B' approaches B as a limit, and V' approaches V as its limit.

But however great the number of the lateral faces,

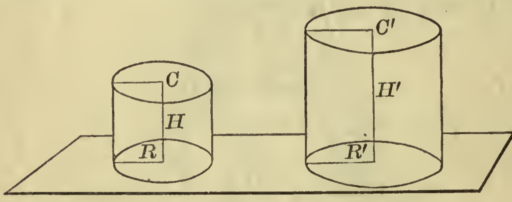
$V' = B' \times H. \quad \therefore V = B \times H.$ § 260

Q. E. D.

650. Cor. *If V denotes the volume, R the radius, H the altitude, of a cylinder of revolution, then the area of the base is πR^2, and $V = \pi R^2 \times H$.*

Proposition XXXIV. Theorem.

651. *The lateral areas, or the total areas, of similar cylinders of revolution are to each other as the squares of their altitudes, or of their radii; and their volumes are to each other as the cubes of their altitudes, or of their radii.*



Let S, S' denote the lateral areas, T, T' the total areas, V, V' the volumes, H, H' the altitudes, R, R' the radii, of two similar cylinders of revolution.

To prove $S : S' = T : T' = H^2 : H'^2 = R^2 : R'^2$,

and $V : V' = H^3 : H'^3 = R^3 : R'^3$.

Proof. Since the generating rectangles are similar,

$$\frac{H}{H'} = \frac{R}{R'} = \frac{H+R}{H'+R'}. \qquad \text{§§ 319, 303}$$

Therefore, by §§ 648, 650,

$$\frac{S}{S'} = \frac{2\pi RH}{2\pi R'H'} = \frac{R}{R'} \times \frac{H}{H'} = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}.$$

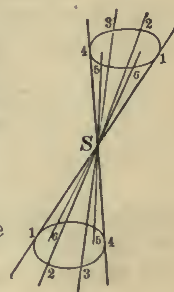
$$\frac{T}{T'} = \frac{2\pi R(H+R)}{2\pi R'(H'+R')} = \frac{R}{R'}\left(\frac{H+R}{H'+R'}\right) = \frac{R^2}{R'^2} = \frac{H^2}{H'^2}.$$

$$\frac{V}{V'} = \frac{\pi R^2 H}{\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{R^3}{R'^3} = \frac{H^3}{H'^3}.$$

Q. E. D.

The Cone.

652. A *conical surface* is the surface generated by a moving straight line called the *generatrix*, passing through a fixed point called the *vertex*, and constantly touching a fixed curve called the *directrix*.



653. The generatrix in any position is called an *element* of the surface. If the generatrix is of indefinite length, the surface consists of two portions, one above and the other below the vertex, which are called the *upper* and *lower nappes*, respectively.

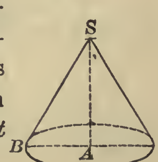
Through a given point in a conical surface one element, and only one, can be drawn.

654. If the directrix is a closed curve, the solid bounded by the conical surface and a plane cutting all its elements is called a *cone*. The conical surface is called the *lateral surface*, and the plane surface the *base*, of the cone. The length of the perpendicular from the vertex to the plane of the base is called the *altitude* of the cone.

655. A *circular cone* is a cone whose base is a circle. The straight line joining the vertex and the centre of the base is called the *axis* of the cone.

If the axis is perpendicular to the base, the cone is called a *right* cone; otherwise, the cone is called an *oblique* cone.

656. A *right circular cone* is a cone whose axis is perpendicular to its base, and is called a *cone of revolution*, because it may be generated by the revolution of a right triangle about one of its legs as an axis. The hypotenuse in any position is an element of the surface, and is called the *slant height* of the cone.



657. *Similar cones of revolution* are cones generated by the revolution of similar right triangles about homologous legs.

658. A *tangent line* to a cone is a line, not an element, which touches the surface of the cone and does not cut it.

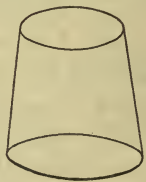
659. A plane which contains an element of the cone and does not cut the surface, is called a *tangent plane.* The element contained by the plane is called the *element of contact.*

660. A pyramid is *inscribed* in a cone when its lateral edges are elements of the cone and its base is inscribed in the base of the cone.

661. A pyramid is *circumscribed* about a cone when its base is circumscribed about the base of the cone and its vertex coincides with the vertex of the cone.

662. A *frustum of a cone* is the portion of a cone included between the base and a section parallel to the base and cutting all the elements.

663. The base of the cone is called the *lower* base of the frustum, and the parallel section the *upper* base.



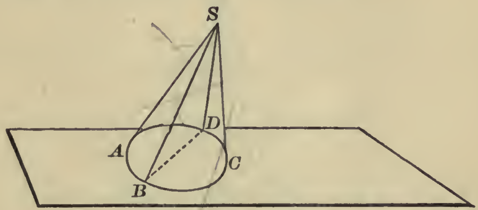
664. The *altitude* of a frustum of a cone is the length of the perpendicular between the planes of its bases.

665. The *lateral surface* of a frustum of a cone is the portion of the lateral surface of the cone included between the bases of the frustum.

666. The *slant height* of a frustum of a cone of revolution is the portion of any element of the cone included between the bases.

PROPOSITION XXXV. THEOREM.

667. *Every section of a cone made by a plane passing through its vertex is a triangle.*



Let a plane pass through the vertex S ***and cut the base in*** BD.

To prove the section SBD *a triangle.*

Proof. Draw the straight lines SB and SD.

Then SB and SD are elements of the surface of the cone, and they lie in the cutting plane, since they have each two points in common with the plane. Hence they are the intersections of the conical surface with the cutting plane.

And BD is a straight line. § 471

Therefore the section SBD is a triangle.

Q. E. D

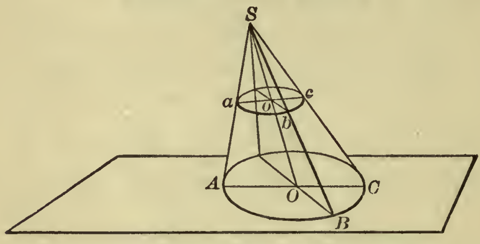
Ex. 510. Show that any lateral face of a pyramid circumscribed about a cone is tangent to the cone.

Ex. 511. The diagonals of a parallelopiped bisect each other.

Ex. 512. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of its three dimensions.

Proposition XXXVI. Theorem.

668. *Every section of a circular cone made by a plane parallel to the base is a circle.*



Let the section *abc* of the circular cone *S-ABC* be parallel to the base.

To prove that abc is a circle.

Proof. Let O be the centre of the base, and let o be the point in which the axis SO pierces the plane of the parallel section.

Through SO and the elements SA, SB, etc., pass planes cutting the base in the radii OA, OB, etc.,

and the section abc in the straight lines oa, ob, etc.

Since abc is $\parallel$ to ABC, oa and ob are $\parallel$ respectively to OA and OB. § 492

Therefore the △ Soa and Sob are similar respectively to the △ SOA and SOB. §§ 106, 321

$$\therefore \frac{oa}{OA} = \left(\frac{So}{SO}\right) = \frac{ob}{OB}.$$

But $OA = OB$. § 211

$$\therefore oa = ob.$$

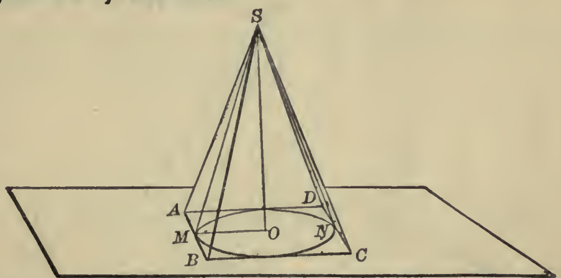
That is, all the straight lines drawn from o to the perimeter of the section are equal.

$\therefore$ the section abc is a $\odot$. Q. E. D.

669. Cor. *The axis of a circular cone passes through the centres of all the sections which are parallel to the base.*

Proposition XXXVII. Theorem.

670. *The lateral area of a cone of revolution is equal to one-half the product of the slant height by the circumference of the base.*



Let S denote the lateral area, C the circumference of the base, and L the slant height, of the cone.

To prove $S = \frac{1}{2} C \times L$.

Proof. Circumscribe about the base any regular polygon $ABCD$, and upon this polygon as a base construct the regular pyramid S-$ABCD$ circumscribed about the cone.

If the lateral area of this pyramid is s, the perimeter p, the slant height L, $s = \frac{1}{2} p \times L$. § 596

Let the number of the lateral faces of the circumscribed pyramid be indefinitely increased, the new edges continually bisecting the arcs of the base. Then p and s approach C and S respectively as their limits.

But however great the number of lateral faces of the pyramid,
$$s = \tfrac{1}{2} p \times L.$$
$$\therefore S = \tfrac{1}{2} C \times L. \qquad \S\ 260$$

Q. E. D.

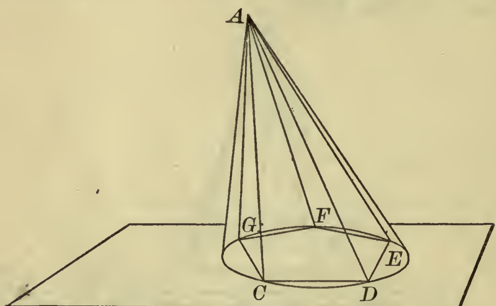
671. Cor. Since $C = 2\pi R$, § 419

$$S = \tfrac{1}{2}(2\pi R \times L) = \pi RL.$$

The total area $T = \pi RL + \pi R^2 = \pi R(L + R)$.

PROPOSITION XXXVIII. THEOREM.

672. *The volume of any cone is equal to the product of one-third of its base by its altitude.*



Let V denote the volume, B the base, and H the altitude of the cone.

To prove $V = \frac{1}{3} B \times H.$

Proof. Let the volume of an inscribed pyramid $A\text{-}CDEFG$ be denoted by V', its base by B' and its altitude by H.

Then $V' = \frac{1}{3} B' \times H.$ § 604

Let the number of lateral faces of the inscribed pyramid be indefinitely increased, the new edges continually dividing the arcs in the base of the cone. Then V' approaches V as its limit, and B' approaches B as its limit.

But however great the number of lateral faces of the pyramid,

$$V' = \frac{1}{3} B' \times H.$$

$$\therefore V = \frac{1}{3} B \times H. \qquad \S\ 260$$

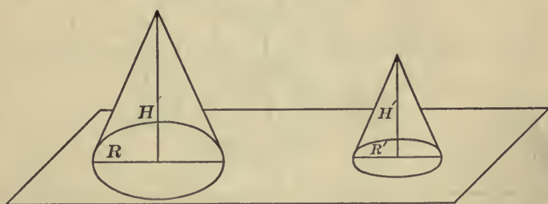
Q. E. D.

673. COR. *If the cone is a cone of revolution, and R is the radius of the base,* $B = \pi R^2$ (§ 425),

and $V = \frac{1}{3} \pi R^2 \times H.$

PROPOSITION XXXIX. THEOREM.

674. *The lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their altitudes, or of their radii; and their volumes are to each other as the cubes of their altitudes, or of their radii.*



Let S and S' denote the lateral areas, T and T' the total areas, V and V' the volumes, H and H' the altitudes, R and R' the radii, L and L' the slant heights, of two similar cones of revolution.

To prove $S : S' = T : T' = H^2 : H'^2 = R^2 : R'^2 = L^2 : L'^2,$

and $V : V' = H^3 : H'^3 = R^3 : R'^3 = L^3 : L'^3.$

Proof. Since the generating triangles are similar,

$$\frac{H}{H'} = \frac{R}{R'} = \frac{L}{L'} = \frac{L+R}{L'+R'}. \qquad \text{§§ 319, 303}$$

Therefore, by §§ 671, 673,

$$\frac{S}{S'} = \frac{\pi RL}{\pi R'L'} = \frac{R}{R'} \times \frac{L}{L'} = \frac{R^2}{R'^2} = \frac{L^2}{L'^2} = \frac{H^2}{H'^2},$$

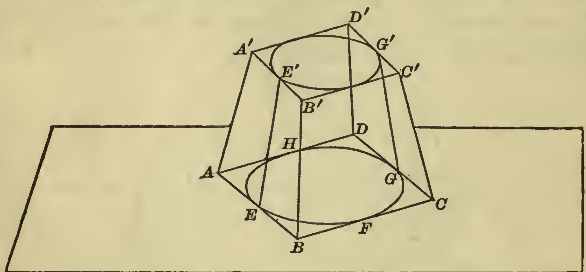
$$\frac{T}{T'} = \frac{\pi R(L+R)}{\pi R'(L'+R')} = \frac{R}{R'} \times \frac{L+R}{L'+R'} = \frac{R^2}{R'^2} = \frac{L^2}{L'^2} = \frac{H^2}{H'^2},$$

and $$\frac{V}{V'} = \frac{\frac{1}{3}\pi R^2 H}{\frac{1}{3}\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{R^3}{R'^3} = \frac{H^3}{H'^3} = \frac{L^3}{L'^3}.$$

Q. E. D.

PROPOSITION XL. THEOREM.

675. *The lateral area of the frustum of a cone of revolution is equal to one-half the sum of the circumferences of its bases multiplied by the slant height.*



Let S denote the lateral area, C and c the circumferences of its bases, R and r their radii, and L the slant height.

To prove $S = \frac{1}{2}(C + c) \times L.$

Circumscribe about the frustum of the cone the frustum of the regular pyramid $ABCD\text{-}A'B'C'D'$, and denote the lateral area of this frustum by s, the perimeters of its lower and upper bases by P and p respectively, and its slant height by L.

Then $s = \frac{1}{2}(P + p) \times L.$ § 597

Let the number of lateral faces be indefinitely increased, the new elements constantly bisecting the arcs of the bases. Then P and p approach C and c, respectively, as their limits.

But, however great the number of lateral faces of the frustum of the pyramid,

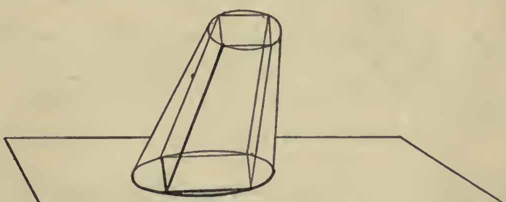
$s = \frac{1}{2}(P + p) \times L. \quad \therefore S = \frac{1}{2}(C + c) \times L.$ § 260

Q. E. D.

676. COR. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equidistant from its bases multiplied by its slant height.*

Proposition XLI. Theorem.

677. *The volume of a frustum of a cone is equivalent to the sum of the volumes of three cones whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.*



Let V denote the volume of the frustum, B its lower base, b its upper base, and H its altitude.

To prove $V = \frac{1}{3} H(B + b + \sqrt{B \times b})$.

Proof. Let V' denote the volume, B' and b' the lower and upper bases, and H the altitude, of an inscribed frustum of a pyramid.

Then $V' = \frac{1}{3} H(B' + b' + \sqrt{B' \times b'})$. § 610

Let the number of lateral faces of the inscribed frustum be indefinitely increased, the new edges continually dividing the arcs in the bases of the frustum of the cone. Then, however great the number of lateral faces of the frustum of the pyramid,

$$V' = \frac{1}{3} H(B' + b' + \sqrt{B \times b'}).$$

$$\therefore V = \frac{1}{3} H(B + b + \sqrt{B \times b}).$$ § 260

Q. E. D.

678. Cor. *If the frustum is that of a cone of revolution, and R and r are the radii of its bases, we have* $B = \pi R^2$, $b = \pi r^2$, *and* $\sqrt{B \times b} = \pi Rr$.

$$\therefore V = \frac{1}{3} \pi H(R^2 + r^2 + Rr).$$

$2\pi r' \times \frac{x}{2} = \frac{1}{2}\, 2\pi r$

Numerical Exercises.

The Pyramid.

Find the volume in cubic feet of a regular pyramid:

513. When its base is a square, each side measuring 3 feet 4 inches, and its height is 9 feet.

514. When its base is an equilateral triangle, each side measuring 4 feet, and its height is 15 feet.

515. When its base is a regular hexagon, each side measuring 6 feet, and its height is 30 feet.

Find the total surface in square feet of a regular pyramid:

516. When each side of its square base is 8 feet, and the slant height is 20 feet.

517. When each side of its triangular base is 6 feet, and the slant height is 18 feet.

518. When each side of its square base is 26 feet, and the perpendicular height is 84 feet.

Find the height in feet of a pyramid when:

519. The volume is 26 cubic feet 936 cubic inches, and each side of its square base is 3 feet 6 inches.

520. The volume is 20 cubic feet, and the sides of its triangular base are 5 feet, 4 feet, and 3 feet.

521. The base edge of a regular pyramid with a square base measures 40 feet, the lateral edge 101 feet; find its volume in cubic feet.

522. Find the volume of a regular pyramid whose slant height is 12 feet, and whose base is an equilateral triangle inscribed in a circle having a radius of 10 feet.

523. Having given the base edge a, and the total surface T, of a regular pyramid with a square base, find the volume V.

524. The base edge of a regular pyramid whose base is a square is a, the total surface T; find the height of the pyramid.

525. The eight edges of a regular pyramid with a square base are equal in length, and the total surface is T; find the length of one edge.

526. Find the base edge a of a regular pyramid with a square base, having given the height h and the total surface T.

Cylinders and Cones.

527. If the total surface of a right circular cylinder closed at both ends is a, and the radius of the base is r, what is the height of the cylinder?

528. If the lateral surface of a right circular cylinder is a, and the volume is b, find the radius of the base and the height.

529. How many cubic yards of earth must be removed in constructing a tunnel 100 yards long, whose section is a semicircle with a radius of 10 feet?

530. If the diameter of a well is 7 feet, and the water is 10 feet deep, how many gallons of water are there, reckoning $7\frac{1}{2}$ gallons to the cubic foot?

531. When a body is placed under water in a right circular cylinder 60 centimeters in diameter, the level of the water rises 30 centimeters; find the volume of the body.

532. If the circumference of the base of a right circular cylinder is c, and the height h, find the volume V.

533. Having given the total surface T of a right circular cylinder, in which the height is equal to the diameter of the base, find the volume V.

534. If the circumference of the base of a right circular cylinder is c, and the total surface is T, find the volume V.

535. The slant height of a right circular cone is 2 feet. At what distance from the vertex must the slant height be cut by a plane parallel to the base, in order that the lateral surface may be divided into two equivalent parts?

536. The height of a right circular cone is equal to the diameter of its base; find the ratio of the area of the base to the lateral surface.

537. What length of canvas $\frac{3}{4}$ of a yard wide is required to make a conical tent 12 feet in diameter and 8 feet high?

538. The circumference of the base of a circular cone is $12\frac{1}{2}$ feet, and its height $8\frac{1}{4}$ feet; find its volume.

539. Given the total surface T of a right circular cone, and the radius r of the base; find the volume V.

540. Given the total surface T of a right circular cone, and the lateral surface S; find the volume V.

533 =

Frustums of Pyramids and Cones.

541. How many square feet of tin will be required to make a funnel if the diameters of the top and bottom are to be 28 inches and 14 inches respectively, and the height 24 inches?

542. Find the expense of polishing the curved surface of a marble column in the shape of the frustum of a right cone whose slant height is 12 feet, and the radii of the circular ends are 3 feet 6 inches and 2 feet 4 inches respectively, at 60 cents a square foot.

543. The slant height of the frustum of a regular square pyramid is 20 feet, the length of each side of its base 40 feet, of each side of its top 16 feet; find its volume.

544. If the bases of the frustum of a pyramid are two regular hexagons whose sides are 1 foot and 2 feet respectively, and the volume of the frustum is 12 cubic feet: find its height.

545. The frustum of a right circular cone is 14 feet high, and has a volume of 924 cubic feet. Find the radii of its bases if their sum is 9 feet.

546. From a right circular cone whose slant height is 30 feet, and circumference of whose base is 10 feet, there is cut off by a plane parallel to the base a cone whose slant height is 6 feet. Find the convex surface and the volume of the frustum.

547. Find the difference between the volume of the frustum of a pyramid whose bases are squares, measuring 8 feet and 6 feet respectively on a side, and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them.

548. A Dutch windmill in the shape of the frustum of a right cone is 12 meters high. The outer diameters at the bottom and the top are 16 meters and 12 meters, the inner diameters 12 meters and 10 meters, respectively. How many cubic meters of stone were required to build it?

549. The chimney of a factory has the shape of a frustum of a regular pyramid. Its height is 180 feet, and its upper and lower bases are squares whose sides are 10 feet and 16 feet respectively. The flue is throughout a square whose side is 7 feet. How many cubic feet of material does the chimney contain?

550. Find the volume V of the frustum of a cone of revolution, having given the slant height a, the height h, and the convex surface S.

$V = \frac{h}{\ldots}\{3S^2 + a^2\pi^2(a^2 - h^2)\}$

EQUIVALENT SOLIDS.

551. A cube whose edge is 12 inches long is transformed into a right prism whose base is a rectangle 16 inches long and 12 inches wide. Find the height of the prism, and the difference between its total surface and the surface of the cube.

552. The dimensions of a rectangular parallelopiped are a, b, c. Find (i.) the height of an equivalent right circular cylinder having a for the radius of its base; (ii.) the height of an equivalent right circular cone having a for the radius of its base.

553. A regular pyramid 12 feet high is transformed into a regular prism with an equivalent base; what is the height of the prism?

554. The diameter of a cylinder is 14 feet, and its height is 8 feet; find the height of an equivalent right prism, the base of which is a square with a side 4 feet long.

555. If one edge of a cube is a, what is the height h of an equivalent right circular cylinder whose diameter is b?

556. The heights of two equivalent right circular cylinders are as 4 : 9. The diameter of the first is 6 feet; what is the diameter of the other?

557. A right circular cylinder 6 feet in diameter is equivalent to a right circular cone 7 feet in diameter. If the height of the cone is 8 feet, what is the height of the cylinder?

558. The frustum of a regular four-sided pyramid is 6 feet high, and the sides of its bases are 5 feet and 8 feet respectively. What is the height of an equivalent regular pyramid whose base is a square with a side 12 feet long?

559. The frustum of a cone of revolution is 5 feet high, and the diameters of its bases are 2 feet and 3 feet respectively; find the height of an equivalent right circular cylinder whose base is equal in area to the section of the frustum made by a plane parallel to its bases, and equidistant from the bases.

560. Find the edge of a cube equivalent to a regular tetrahedron whose edge measures 3 inches.

561. Find the edge of a cube equivalent to a regular octahedron whose edge measures 3 inches.

SIMILAR SOLIDS.

562. The dimensions of a trunk are 4 feet, 3 feet, 2 feet. What are the dimensions of a trunk similar in shape that will hold four times as much?

563. By what number must the dimensions of a cylinder be multiplied in order to obtain a similar cylinder (i.) whose surface shall be n times that of the first; (ii.) whose volume shall be n times that of the first?

564. A pyramid is cut by a plane which passes midway between the vertex and the plane of the base. Compare the volumes of the entire pyramid and the pyramid cut off.

565. The height of a regular hexagonal pyramid is 36 feet, and one side of the base is 6 feet. What are the dimensions of a similar pyramid whose volume is $\frac{1}{20}$ that of the first?

566. The length of one of the lateral edges of a pyramid is 4 meters. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts?

567. The length of a lateral edge of a pyramid is a. At what distances from the vertex will this edge be cut by two planes parallel to the base, which divide the pyramid into three equivalent parts?

568. The length of a lateral edge of a pyramid is a. At what distance from the vertex will this edge be cut by a plane parallel to the base, and dividing the pyramid into two parts which are to each other as $3:4$?

569. The volumes of two similar cones are 54 cubic feet and 432 cubic feet. The height of the first is 6 feet; what is the height of the other?

570. In each of two right circular cylinders the diameter is equal to the height. The volume of one is $\frac{3}{4}$ that of the other. What is the ratio of their heights?

571. Find the dimensions of a right circular cylinder $\frac{15}{16}$ as large as a similar cylinder whose height is 20 feet, and diameter 10 feet.

572. The height of a cone of revolution is h, and the radius of its base is r. What are the dimensions of a similar cone three times as large?

573. The height of the frustum of a right cone is $\frac{2}{5}$ the height of the entire cone. Compare the volumes of the frustum and the entire cone.

574. The frustum of a pyramid is 8 feet high, and two homologous edges of its bases are 4 feet and 3 feet respectively. Compare the volume of the frustum and that of the entire pyramid.

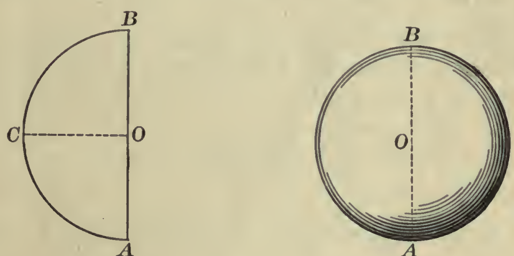
BOOK VIII.

THE SPHERE.

Plane Sections and Tangent Planes.

679. A *sphere* is a solid bounded by a surface every point of which is equally distant from a point called the *centre.*

680. A sphere may be generated by the revolution of a semicircle ACB about its diameter AB as an axis.



681. A *radius* of a sphere is a straight line drawn from its centre to its surface.

682. A *diameter* of a sphere is a straight line passing through the centre and limited by the surface.

Since all the radii of a sphere are equal, and a diameter is equal to two radii, all the diameters of a sphere are equal.

683. A line or plane is *tangent* to a sphere when it has one, and only one, point in common with the surface of the sphere.

684. Two spheres are tangent to each other when their surfaces have one, and only one, point in common.

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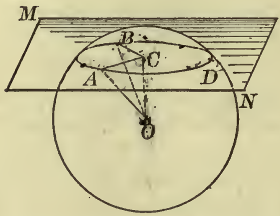
Proposition I. Theorem.

685. *Every section of a sphere made by a plane is a circle.*



Let O be the centre of a sphere, and ABD any section made by a plane.

To prove that the section ABD is a circle.

Proof. Draw the radii OA, OB, to any two points A, B, in the boundary of the section, and draw $OC \perp$ to the section.

In the rt. $\triangle$ OAC, OBC,

$$OC \text{ is common.}$$

Also $$OA = OB,$$

$$\textit{(being radii of the sphere).}$$

$$\therefore \triangle OAC = \triangle OBC, \qquad \S\ 161$$

$$\therefore CA = CB.$$

In like manner any two points in the boundary of the section may be proved to be equally distant from C.

Hence the section ABD is a circle whose centre is C. Q. E. D.

686. Cor. 1. *The line joining the centre of a sphere to the centre of a circle of the sphere is perpendicular to the plane of the circle.*

687. Cor. 2. *Circles of a sphere made by planes equally distant from the centre are equal.* For $\overline{AC}^2 = \overline{AO}^2 - \overline{OC}^2$; and AO and OC are the same for all equally distant circles; therefore AC is the same.

688. Cor. 3. *Of two circles made by planes unequally distant from the centre, the nearer is the larger.* For, in the expression $\overline{AC}^2 = \overline{AO}^2 - \overline{OC}^2$, as OC decreases, AC increases.

689. A *great circle* of a sphere is a section made by a plane which passes through the centre of the sphere.

690. A *small circle* of a sphere is a section made by a plane which does not pass through the centre of the sphere.

691. The *axis* of a circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle. The ends of the axis are called the *poles*.

692. *Parallel circles have the same axis and the same poles.*

693. *All great circles of a sphere are equal.*

694. *Every great circle bisects the sphere.* For the two parts into which the sphere is divided can be so placed that they will coincide; otherwise there would be points on the surface unequally distant from the centre.

695. *Two great circles bisect each other.* For the intersection of their planes passes through the centre, and is a diameter of each circle.

696. *Two great circles whose planes are perpendicular pass through each other's poles;* and conversely.

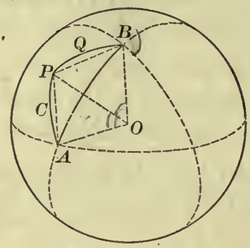
697. *Through two given points on the surface of a sphere an arc of a great circle may always be drawn.* For the two given points together with the centre of the sphere determine the plane of a great circle whose circumference passes through the two given points.

If the two given points are the ends of a diameter, the position of the circle is not determined; for through a diameter an indefinite number of planes may be passed.

698. *Through three given points on the surface of a sphere one circle may be drawn, and only one.* For the three points determine one, and only one, plane.

PROPOSITION II. THEOREM.

699. *The shortest distance on the surface of a sphere between any two points on that surface is the arc, not greater than a semi-circumference, of the great circle which joins them.*



Let AB be the arc of a great circle which joins any two points A and B on the surface of a sphere; and let $ACPQB$ be any other line on the surface between A and B.

To prove $ACPQB > AB$.

Proof. Let P be any point in $ACPQB$.

Let arcs of great circles pass through A, P, and P, B. § 697

Join A, P, and B with the centre of the sphere O.

The ∠s AOB, AOP, and POB are the face ∠s of the trihedral angle whose vertex is at O.

The arcs AB, AP, and PB are measures of these ∠s. § 262

Now $\angle AOP + \angle POB$ is greater than $\angle AOB$, § 539

$\therefore$ arc $AP +$ arc $PB >$ arc AB.

In like manner, joining any point in ACP with A and P, and any point in PQB with P and B, by arcs of great ⓢ, the sum of these arcs will be greater than arc $AP +$ arc PB; and therefore greater than arc AB.

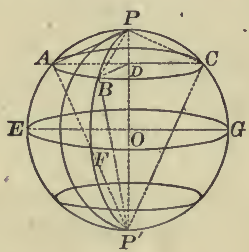
If this process be indefinitely repeated, the sum of the arcs of the great ⓢ will increase and always be greater than AB.

Therefore $ACPQB$, which is the limit of the sum of these arcs, is greater than AB. Q. E. D.

700. By the *distance* between two points on the surface of a sphere is meant the *arc of a great circle* joining them.

Proposition III. Theorem.

701. ***The distances of all points in the circumference of a circle of a sphere from its poles are equal.***



Let P, P' be the poles of the circle ABC, and A, B, C, any points on its circumference.

To prove that the great circle arcs PA, PB, PC are equal.

Proof. The straight lines PA, PB, PC are equal, § 478

Therefore the arcs PA, PB, PC are equal. § 230

In like manner, the great circle arcs $P'A$, $P'B$, $P'C$ may be proved equal. Q. E. D.

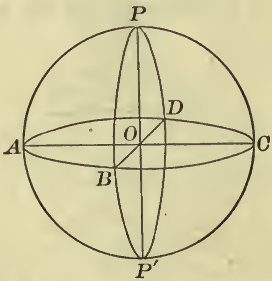
702. The distance from the nearer pole of a circle to any point in the circumference of the circle is called the *polar distance* of the circle.

703. Cor. 1. *The polar distance of a great circle is a quadrant-arc.* For it is the measure of a right angle whose vertex is at the centre of the sphere.

704. Scholium. The distances of all points in the circumference of a circle of a sphere from any point in its axis are equal.

PROPOSITION IV. THEOREM.

705. *A point on the surface of a sphere, which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is a pole of the great circle passing through these points.*



Let the distances PA and PB be quadrants.

To prove P a pole of the great circle which passes through A and B.

Proof. The $\angle$s POA and POB are rt. $\angle$s,

(*because each is measured by an arc equal to a quadrant*).

$\therefore PO$ is $\perp$ to the plane of the $\odot$ ABC, § 472

Hence P is a pole of the $\odot$ ABC. § 691

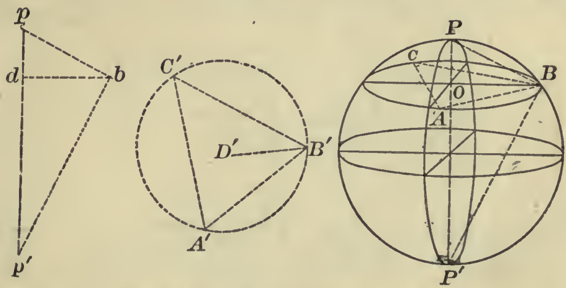
Q. E. D.

706. COR. *The above theorem enables us to describe with the compasses an arc of a great circle through two given points A and B of the surface of a sphere.* For, if with A and B as centres, and an opening of the compasses equal to the *chord* of a quadrant of a great circle, we describe arcs, these arcs will cut at a point P, which will be the pole of the great circle passing through A and B. Then with P as centre, the arc passing through A and B may be described.

In order to make the opening of the compasses equal to the chord of a quadrant of a great circle, the radius or the diameter of the sphere must be given.

Proposition V. Problem.

707. *Given a material sphere to find its radius.*



Let PBP'C represent a material sphere.

It is required to find its diameter.

Construction. From any point P of the given surface, with any opening of the compasses, describe the circumference ABC on the surface. Then the straight line PB is known.

Take any three points A, B, and C in this circumference, and with the compasses measure the chord distances AB, BC, and CA.

Construct the $\triangle\ A'B'C'$, with sides equal respectively to AB, BC, and CA, and circumscribe a $\odot$ about the $\triangle\ A'B'C'$.

The radius $D'B'$ of this $\odot$ is equal to the radius of $\odot\ ABC$.

Construct the rt. $\triangle\ bdp$, having the hypotenuse $bp = BP$, and one side $bd = B'D'$.

Draw $bp' \perp$ to bp, and meeting pd produced in p'.

Then pp' is equal to the diameter of the given sphere.

Proof. Suppose the diameter PP' and the straight line $P'B$ drawn.

The $\triangle$ OBP and bdp are equal. § 161

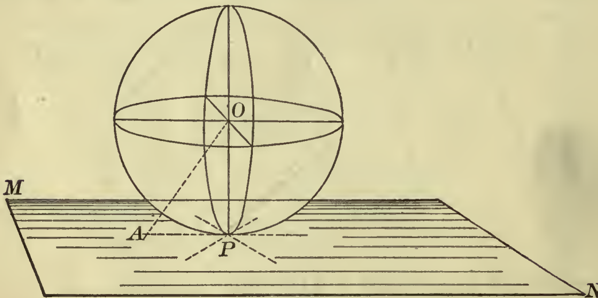
Hence the $\triangle$ PBP' and pbp' are equal. § 149

Therefore $pp' = PP'$.

And $\frac{1}{2} pp'$ is equal to the radius. Q. E. F.

PROPOSITION VI. THEOREM.

708. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*



Let O be the centre of a sphere, and MN a plane perpendicular to the radius OP, at its extremity P

To prove MN tangent to the sphere.

Proof. From O draw any other straight line OA to the plane MN.

$OP < OA$, § 477

(*a $\perp$ is the shortest distance from a point to a plane*).

Therefore the point A is without the sphere.

Similarly we may prove that every point, except P, in the plane MN is without the sphere,

Therefore MN is tangent to the sphere at P. § 683

Q. E. D.

709. COR. 1. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

710. COR. 2. *A straight line tangent to a circle of a sphere lies in a plane tangent to the sphere at the point of contact.* § 473

711. COR. 3. *Any straight line in a tangent plane through the point of contact is tangent to the sphere at that point.*

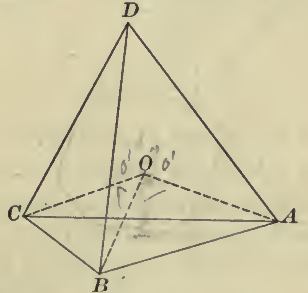
712. COR. 4. *The plane of two straight lines tangent to a sphere at the same point is tangent to the sphere at that point.*

713. A sphere is said to be *inscribed in a polyhedron* when all the faces of the polyhedron are tangent to the sphere.

714. A sphere is said to be *circumscribed about a polyhedron* when all the vertices of the polyhedron lie in the surface of the sphere.

Proposition VII. Theorem.

715. ***A sphere may be inscribed in any given tetrahedron.***



Let ABCD be the given tetrahedron.

To prove that a sphere may be inscribed in ABCD.

Proof. Bisect the dihedral ∠s at the edges AB, BC, and AC by the planes OAB, OBC, and OAC, respectively.

Every point in the plane OAB is equally distant from the faces ABC and ABD. § 525

For a like reason, every point in the plane OBC is equally distant from the faces ABC and DBC; and every point in the plane OAC is equally distant from the faces ABC and ADC.

Therefore O, the common intersection of these three planes, is equally distant from the four faces of the tetrahedron.

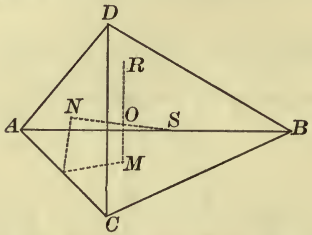
Hence a sphere described with O as a centre, and with the radius equal to the distance from O to any face, will be tangent to each face, and will be inscribed in the tetrahedron. § 713

Q. E. D.

716. Cor. *The six planes which bisect the six dihedral angles of a tetrahedron intersect in the same point.*

Proposition VIII. Theorem.

717. *A sphere may be circumscribed about any given tetrahedron.*



Let ABCD be the given tetrahedron.

To prove that a sphere may be circumscribed about ABCD.

Proof. Let M, N, respectively be the centres of the circles circumscribed about the faces ABC, ACD.

Let also MR be $\perp$ to face ABC, $NS \perp$ to face ACD.

MR is the locus of points equidistant from A, B, C,

and NS is the locus of points equidistant from A, C, D. § 480

Also MR and NS lie in the same plane.

For, if a plane $\perp$ to AC be passed through its middle point, this plane will contain all points equidistant from A and C. § 482

$\therefore$ MR and NS must lie in this plane.

Also MR and NS, being $\perp$ to planes which are not $\parallel$, cannot be $\parallel$, and must therefore meet at some point O.

$\therefore$ O is equidistant from A, B, C, and D,

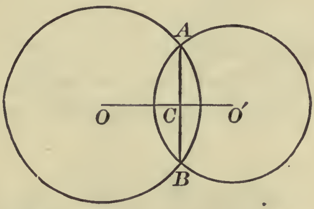
and a spherical surface whose centre is O, and radius OA, will pass through the points A, B, C, and D. Q. E. D

718. Cor. 1. *The four perpendiculars erected at the centres of the faces of a tetrahedron meet at the same point.*

719. Cor. 2. *The six planes perpendicular to the edges of a tetrahedron at their middle points intersect at the same point.*

Proposition IX. Theorem.

720. *The intersection of two spherical surfaces is the circumference of a circle whose plane is perpendicular to the line joining the centres of the surfaces and whose centre is in that line.*



Let *O*, *O′* be the centres of the spherical surfaces, and let a plane passing through *O*, *O′* cut the sphere in great circles whose circumferences intersect each other in the points *A* and *B*.

To prove that the spherical surfaces intersect in the circumference of a circle whose plane is perpendicular to OO', and whose centre is the point C where AB meets OO'.

Proof. The common chord AB is ⊥ to OO' and bisected at C, § 249

(*when two circumferences intersect each other, the line joining their centres is ⊥ to the common chord at its middle point*).

If the plane of the two great circles revolve about OO', their circumferences will generate the two spherical surfaces, and the point A will describe the line of intersection of the surfaces.

But during the revolution AC will remain constant in length and ⊥ to OO'.

Therefore the line of intersection described by the point A will be the circumference of a circle whose centre is C and whose plane is ⊥ to OO'. § 473

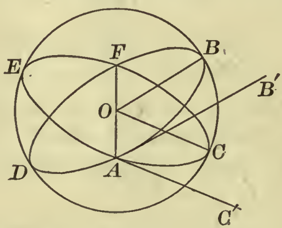
Q. E. D.

FIGURES ON THE SURFACE OF A SPHERE.

721. The *angle of two curves* passing through the same point is the angle formed by the two straight lines tangent to the curves at that point. If the two curves are arcs of great circles, the angle is called a *spherical angle.*

PROPOSITION X. THEOREM.

722. *A spherical angle is measured by the arc of a great circle described from its vertex as a pole and included between its sides (produced if necessary).*



Let AB, AC be arcs of great circles intersecting at A; AB' and AC', the tangents to these arcs at A; BC an arc of a great circle described from A as a pole and included between AB and AC.

To prove that the spherical $\angle BAC$ *is measured by arc BC.*

Proof. Draw the radii OA, OB, OC.

In the plane AOB, AB' is $\perp$ to AO, § 240

and OB is $\perp$ to AO.

$\therefore AB'$ is $\parallel$ to OB. § 100

Similarly, AC' is $\parallel$ to OC.

$\therefore \angle B'AC' = \angle BOC$. § 498

But $\angle BOC$ is measured by arc BC. § 262

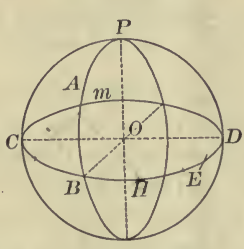
$\therefore \angle B'AC'$ is measured by arc BC.

$\therefore \angle BAC$ is measured by arc BC. Q. E. D.

723. COR. *A spherical angle has the same measure as the dihedral angle formed by the planes of the two circles.*

Proposition XI. Problem.

724. *To describe an arc of a great circle through a given point perpendicular to a given arc of a great circle.*



Let A be a point on the surface of a sphere, CHD an arc of a great circle, P its pole.

To describe an arc of a great circle through A perpendicular to CHD.

Construction. From A as a pole describe an arc of a great circle cutting CHD at E.

From E as a pole describe the arc AB through A.

Then AB is the arc required.

Proof. The arc AB is the arc of a great circle. and E is its pole by construction. § 703

The point E is at the distance of a quadrant from P. § 703

Therefore the arc AB produced will pass through P.

And since the spherical $\angle PBE$ is measured by an arc of a great circle extending from P to E, § 722

the $\angle ABD$ is a right angle.

Therefore the arc AB is $\perp$ to the arc CHD. Q. E. D.

Ex. 575. Every point in a great circle which bisects a given arc of a great circle at right angles, is equidistant from the extremities of the given arc.

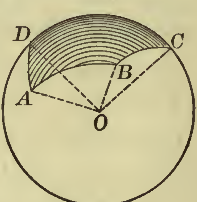
725. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of great circles.

The bounding arcs are the *sides* of the polygon; the angles which they form are the *angles* of the polygon; their points of intersection are the *vertices* of the polygon.

The values of the sides of a spherical polygon are usually expressed in degrees, minutes, and seconds.

726. The planes of the sides of a spherical polygon form a polyhedral angle whose vertex is the centre of the sphere, whose face angles are measured by the sides of the polygon, and whose dihedral angles have the same numerical measure as the angles of the polygon.

Thus, the planes of the sides of the polygon $ABCD$ form the polyhedral angle O-$ABCD$. The face angles AOB, BOC, etc., are measured by the sides AB, BC, etc., of the polygon. The dihedral angle whose edge is OA has the same measure as the spherical angle BAD, etc.



Hence, *from any property of polyhedral angles we may infer an analogous property of spherical polygons;* and conversely.

727. A spherical polygon is *convex* if the corresponding polyhedral angle is convex (§ 534). Every spherical polygon is to be assumed convex unless otherwise stated.

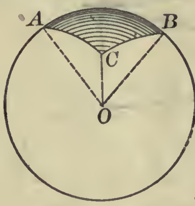
728. A *diagonal* of a spherical polygon is an arc of a great circle connecting any two vertices which are not adjacent.

729. A *spherical triangle* is a spherical polygon of three sides; like a plane triangle, it may be *right* or *oblique*, *equilateral*, *isosceles*, or *scalene*.

730. Two spherical polygons are *equal* if they can be applied, the one to the other, so as to coincide.

Proposition XII. Theorem.

731. *Each side of a spherical triangle is less than the sum of the other two sides.*



Let ABC ***be a spherical triangle,*** AB ***the largest side.***

To prove $AB < AC + BC$.

Proof. In the corresponding trihedral angle $O\text{-}ABC$,

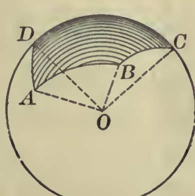
$\angle AOB$ is less than $\angle AOC + \angle BOC$. § 539

$\therefore AB < AC + BC$. § 726

Q. E. D.

Proposition XIII. Theorem.

732. *The sum of the sides of a spherical polygon is less than 360°.*



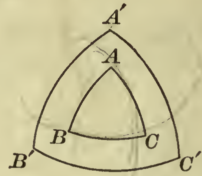
Let $ABCD$ ***be a spherical polygon.***

To prove $AB + BC + CD + DA < 360°$.

Proof. In the corresponding polyhedral angle $O\text{-}ABCD$, the sum of all the face angles is less than 360°. § 540

$\therefore AB + BC + CD + DA < 360°$. Q. E. D.

733. If, from the vertices of a spherical triangle as poles, arcs of great circles are described, a spherical triangle is formed, which is called the *polar triangle* of the first. Thus, if A, B, C are the poles of the arcs of the great circles $B'C'$, $A'C'$, $A'B'$, respectively, then $A'B'C'$ is the polar triangle of ABC.

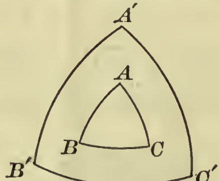


If, with A, B, C as poles, entire great circles instead of arcs are described, these circles will divide the surface of the sphere into *eight* spherical triangles.

Of these eight triangles, that one is the polar of ABC whose vertex A', corresponding to A, lies on the same side of BC as the vertex A; and similarly with the other vertices.

PROPOSITION XIV. THEOREM.

734. *If $A'B'C'$ is the polar triangle of ABC, then, reciprocally, ABC is the polar triangle of $A'B'C'$.*



Let $A'B'C'$ ***be the polar triangle of*** ABC.

To prove that ABC is the polar triangle of $A'B'C'$.

Proof. Since A is the pole of $B'C'$, § 733

$\therefore B'$ is at a quadrant's distance from A. § 703

Similarly, since C is the pole of $A'B'$,

$\therefore B'$ is at a quadrant's distance from C.

$\therefore B'$ is the pole of the arc AC. § 705

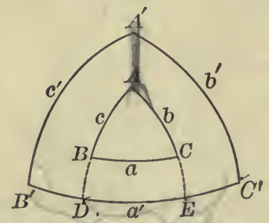
Similarly, A' is the pole of BC, and C' the pole of AB.

$\therefore ABC$ is the polar triangle of $A'B'C'$. § 733

Q. E. D.

PROPOSITION XV. THEOREM.

735. *In two polar triangles each angle of the one is the supplement of the opposite side in the other.*



Let ABC, A'B'C' be two polar triangles; then let the letter at the vertex of each angle denote its value in angle-degrees, and the corresponding small letters the values of the opposite sides in arc-degrees.

To prove $A + a' = 180°$, $B + b' = 180°$, $C + c' = 180°$.

$A' + a = 180°$, $B' + b = 180°$, $C' + c = 180°$.

Proof. Produce the arcs AB, AC until they meet $B'C'$ at the points D, E, respectively.

Since B' is the pole of AE, $B'E = 90°$.

Since C' is the pole of AD, $C'D = 90°$.

Adding, we have $B'E + C'D = 180°$.

That is, $B'D + DE + C'D = 180°$.

Or $DE + B'C' = 180°$.

But $B'C' = a'$.

Also DE measures $\angle A$, § 722

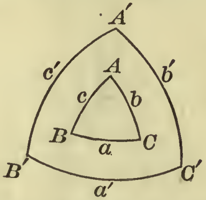
$\therefore A + a' = 180°$.

In a similar way all the other relations are proved. Q. E. D.

736. SCHOLIUM. Two polar triangles are sometimes called *supplemental* triangles.

PROPOSITION XVI. THEOREM.

737. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*



Let ABC be a spherical triangle, and let A, B, C denote the values of its angles, and a', b', c', respectively, the values of the opposite sides in the polar triangle $A'B'C'$.

To prove $A+B+C>180°$ *and* $<540°$.

Proof. Since the $\triangle$ ABC, $A'B'C'$, are polar $\triangle$,

$$A+a'=180°,\ B+b'=180°,\ C+c'=180°. \qquad \S\ 735$$

By addition, $A+B+C+a'+b'+c'=540°$.

$$\therefore A+B+C=540°-(a'+b'+c').$$

Now $a'+b'+c'$ is less than 360°, § 732

$$\therefore A+B+C=540°-\text{some number less than } 360°.$$

$$\therefore A+B+C>180°.$$

And since $a'+b'+c'$ is greater than 0°,

$$\therefore A+B+C<540°.$$

Q. E. D.

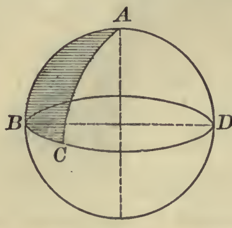
738. COR. *A spherical triangle may have two, or even three, right angles; and it may have two, or even three, obtuse angles.*

739. A spherical triangle having two right angles is called a *bi-rectangular* triangle; and a spherical triangle having three right angles is called a *tri-rectangular* triangle.

740. The difference between the sum of the angles of a spherical triangle and 180° is called the *spherical excess* of the triangle.

PROPOSITION XVII. THEOREM.

741. *In a bi-rectangular spherical triangle the sides opposite the right angles are quadrants, and the side opposite the third angle measures that angle.*



Let ABC be a bi-rectangular spherical triangle, with the angles at B and C right angles.

To prove that AB and AC are quadrants, and that $\angle A$ is measured by BC.

Proof. Since the $\angle$s B and C are right angles, the planes of the arcs AB, AC are $\perp$ to the plane of the arc BC. § 723

$\therefore$ AB and AC must each pass through the pole of BC, § 696
(*two great circles whose planes are $\perp$ pass through each other's poles*).

$\therefore$ A is the pole of BC.

$\therefore$ AB and AC are quadrants, § 703

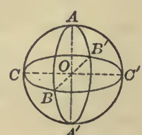
and $\angle A$ is measured by the arc BC. § 722

Q. E. D.

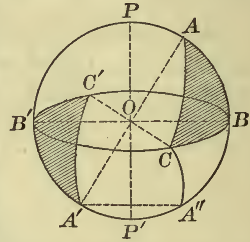
742. COR. 1. *If two sides of a spherical triangle are quadrants, the third side measures the opposite angle.*

743. COR. 2. *Each side of a tri-rectangular spherical triangle is a quadrant.*

744. COR. 3. *Three planes passed through the centre of a sphere, each perpendicular to the other two planes, divide the surface of the sphere into eight tri-rectangular triangles.*



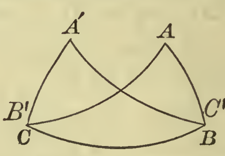
745. If through the centre O of a sphere three diameters AA', BB', CC' are drawn, and the points A, B, C are joined by arcs of great circles, and also the points A', B', C', the two spherical triangles ABC and $A'B'C'$ are called *symmetrical* spherical triangles.



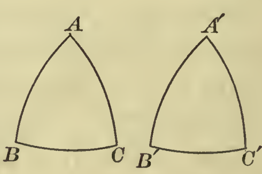
The corresponding trihedral angles are also symmetrical. § 538

In the same way we may form two symmetrical polygons of any number of sides. And after they are formed they may be placed in any positions upon the surface of the sphere.

746. Two symmetrical triangles are mutually equilateral and equiangular; yet in general they cannot be made to coincide by superposition. If in the above figure, the hemisphere below the great circle $BCB'C'$ be revolved about its axis through half a revolution, the triangle $A'B'C'$ will take the position $A''BC$, and it will now be quite evident that the triangles cannot be made to coincide. If the triangles are placed so that $B'C'$ coincides with CB, and A and A' lie on the same side of BC, it will be seen that the equal parts of the two triangles occur in *reverse* order.



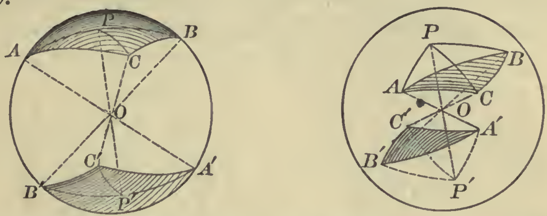
747. If, however, $AB = AC$, and $A'B' = A'C'$; that is, if the two symmetrical triangles are *isosceles*, then, because AB, AC, $A'B'$, $A'C'$, are all equal, and the angles A and A' are equal, being opposite dihedral angles (§ 745), the two triangles can be made to coincide; in other words,



If two symmetrical spherical triangles are isosceles, they are superposable, and therefore equal.

Proposition XVIII. Theorem.

748. *Two symmetrical spherical triangles are equivalent.*



Let ABC, A′B′C′ be two symmetrical spherical triangles with their homologous vertices diametrically opposite to each other.

To prove that the triangles ABC, A′B′C′ are equivalent.

Proof. Let P be the pole of a small circle passing through the points A, B, C, and let POP' be a diameter.

Draw the great circle arcs PA, PB, PC, $P'A'$, $P'B'$, $P'C'$.

$$PA = PB = PC. \qquad \S\ 701$$

And since $P'A' = PA,\ P'B' = PB,\ P'C' = PC,$ § 746

$$\therefore P'A' = P'B' = P'C'.$$

The two symmetrical $\triangle$s PAC, $P'A'C'$ are isosceles.

$$\therefore \triangle PAC = \triangle P'A'C'. \qquad \S\ 747$$

Similarly, $\triangle PAB = \triangle P'A'B'$,

and $\triangle PBC = \triangle P'B'C'$.

Now $\triangle ABC \bumpeq \triangle PAC + \triangle PAB + \triangle PBC$,

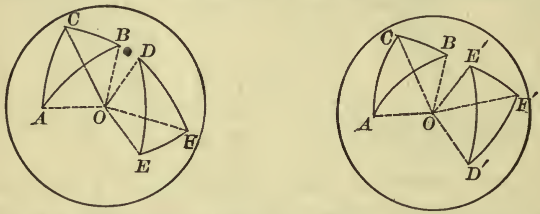
and $\triangle A'B'C' \bumpeq \triangle P'A'C' + \triangle P'A'B' + \triangle P'B'C'$.

$$\therefore \triangle ABC \bumpeq \triangle A'B'C'. \qquad \text{Q. E. D.}$$

If the pole P should fall without the $\triangle ABC$, then P' would fall without $\triangle A'B'C'$, and each triangle would be equivalent to the sum of two isosceles triangles diminished by the third; so that the result would be the same as before.

PROPOSITION XIX. THEOREM.

749. *Two triangles on the same sphere, or equal spheres, are equal or equivalent, if two sides and the included angle of the one are respectively equal to two sides and the included angle of the other.*



I. *In the triangles ABC and DEF let angle A equal angle D, and the sides AB and AC equal respectively the sides DE and DF; and let the parts of the two triangles be arranged in the same order.*

To prove triangles ABC and DEF equal.

Proof. $\triangle ABC$ can be applied to $\triangle DEF$, as in the corresponding case of plane $\triangle$s, and will coincide with it. § 150

II. *In the triangles ABC and D′E′F′ let angle A equal angle D′, and the sides AB and AC equal respectively the sides D′E′ and D′F′; and let the parts of the two triangles be arranged in reverse order.*

To prove triangles ABC and D′E′F′ equivalent.

Proof. Let the $\triangle DEF$ upon the same or an equal sphere be symmetrical with respect to the $\triangle D'E'F'$.

Then $\triangle DEF$ has its $\angle$s and sides equal respectively to those of the $\triangle D'E'F'$.

Also in the $\triangle$s ABC and DEF

$$\angle A = \angle D, \quad AB = DE, \quad AC = DF,$$

and the parts are arranged in the same order.

$$\therefore \triangle ABC = \triangle DEF. \qquad \text{Case I.}$$

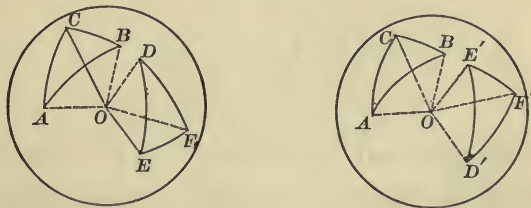
But $$\triangle D'E'F' \approx \triangle DEF, \qquad \text{§ 748}$$

$$\therefore \triangle ABC \approx \triangle D'E'F'.$$

Q. E. D.

PROPOSITION XX. THEOREM.

750. *Two triangles on the same sphere, or equal spheres, are equal or equivalent, if a side and two adjacent angles of the one are equal respectively to a side and two adjacent angles of the other.*



Proof. One of the $\triangle$s may be applied to the other, or to its symmetrical $\triangle$, as in the corresponding case of plane $\triangle$s. § 147

Q. E. D.

PROPOSITION XXI. THEOREM.

751. *Two mutually equilateral triangles on the same sphere, or equal spheres, are mutually equiangular, and are equal or equivalent.*

Proof. The face $\angle$s of the corresponding trihedral $\angle$s at the centre of the sphere are equal respectively,

(*since they are measured by equal sides of the* $\triangle$s).

Therefore the corresponding dihedral $\angle$s are equal. § 542

Hence the $\angle$s of the spherical $\triangle$s are respectively equal.

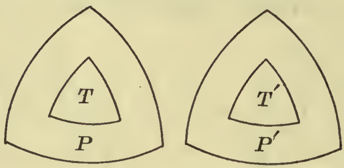
Therefore the $\triangle$s are either equal, or symmetrical and equivalent, according as their equal sides are arranged in the same or reverse order.

Q. E. D.

Ex. 576. The radius of a sphere is 4 inches. From any point on the surface as a pole a circle is described upon the sphere with an opening of the compasses equal to 3 inches. Find the area of this circle.

PROPOSITION XXII. THEOREM.

752. *Two mutually equiangular triangles, on the same sphere, or equal spheres, are mutually equilateral, and are either equal or equivalent.*



Let the spherical triangles T and T' be mutually equiangular.

To prove triangles T and T' mutually equilateral, and equal or equivalent.

Proof. Let ⩓ P and P' be the polar ⩓ of the ⩓ T and T', respectively.

The ⩓ P and P' are mutually equilateral, because in two polar ⩓ each side of the one is the supplement of the angle lying opposite to it in the other. § 735

∴ ⩓ P and P' are mutually equiangular, because two mutually equilateral ⩓ on equal spheres are mutually equiangular. § 751

∴ ⩓ T and T' are mutually equilateral.

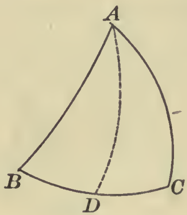
Hence ⩓ T and T' are either equal, or symmetrical and equivalent, because two mutually equilateral ⩓ on equal spheres are either equal, or symmetrical and equivalent. § 751

Q. E. D.

REMARK. The statement that mutually equiangular spherical triangles are mutually equilateral, and equal, or equivalent, is true only when limited to the same sphere, or equal spheres. But when the spheres are unequal, the spherical triangles are unequal; and the ratio of their homologous sides is equal to the ratio of the radii of the spheres on which they are situated. (§ 427.)

Proposition XXIII. Theorem.

753. *In an isosceles spherical triangle, the angles opposite the equal sides are equal.*



In the spherical triangle ABC, let AB equal AC.

To prove $\angle B = \angle C$.

Proof. Draw arc AD of a great circle, from the vertex A to the middle of the base BC.

Then △ ABD and ACD are mutually equilateral.

$\therefore$ △ ABD and ACD are mutually equiangular, § 751
(*two mutually equilateral △ on the same sphere are mutually equiangular*).

$\therefore \angle B = \angle C$,
(*since they are homologous ∠ of symmetrical △*).

Q. E. D.

754. Cor. *The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle of the base bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetrical triangles.*

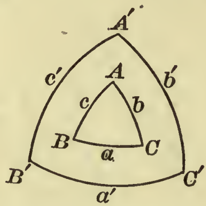
Ex. 577. At a given point in a given arc of a great circle, to construct a spherical angle equal to a given spherical angle.

Ex. 578. To inscribe a circle in a given spherical triangle.

Ex. 579. To circumscribe a circle about a given spherical triangle.

Proposition XXIV. Theorem.

755. *If two angles of a spherical triangle are equal, the sides opposite these angles are equal, and the triangle is isosceles.*



In the spherical triangle ABC, let angle B equal angle C.

To prove $AC = AB$.

Proof. Let the $\triangle$ $A'B'C'$ be the polar $\triangle$ of the $\triangle$ ABC.

By hypothesis $\angle B = \angle C$,

$\therefore A'C' = A'B'$, § 735

in two polar $\triangle$*s, each side of one is the supplement of the* $\angle$ *lying opposite to it in the other).*

$\therefore \angle B' = \angle C'$. § 753

$\therefore AC = AB$. § 735

Q. E. D.

Ex. 580. Given a spherical triangle whose sides are 60°, 80°, and 100°; find the angles of its polar triangle. 120, 100, 80.

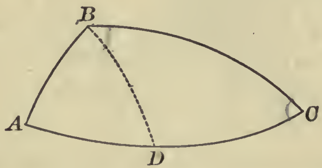
Ex. 581. Given a spherical triangle whose angles are 70°, 75°, and 95°; find the sides of its polar triangle. 110, 105, 85.

Ex. 582. Given two mutually equiangular triangles on spheres whose radii are 12 inches and 20 inches respectively; find the ratio of two homologous sides of these triangles. (See note, page 362.)

$\frac{R}{r} = \frac{12}{20} = \frac{3}{5}$

PROPOSITION XXV. THEOREM.

756. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side is opposite the greater angle;* CONVERSELY, *if two sides are unequal, the angles opposite are unequal, and the greater angle is opposite the greater side.*



I. ***In the triangle ABC, let the angle ABC be greater than the angle ACB.***

To prove $AC > AB$.

Proof. Draw the arc BD of a great circle, making $\angle CBD$ equal $\angle ACB$.

Then $DC = DB$, § 755

Now $AD + DB > AB$, § 731

$\therefore AD + DC > AB$, or $AC > AB$.

II. ***Let AC be greater than AB.***

To prove $\angle ABC$ *greater than* $\angle ACB$.

Proof. The $\angle ABC$ must be equal to, less than, or greater than the $\angle ACB$.

If $\angle ABC = \angle C$, then $AC = AB$, § 755

and if $\angle ABC$ is less than $\angle C$, then $AC < AB$. Case I.

But both of these conclusions are contrary to the hypothesis.

$\therefore \angle ABC$ is greater than $\angle C$. Q. E. D.

MEASUREMENT OF SPHERICAL SURFACES.

757. A *zone* is a portion of the surface of a sphere included between two parallel planes.

The circumferences of the sections made by the planes are called the *bases* of the zone, and the distance between the planes is its *altitude.*

758. A *zone of one base* is a zone one of whose bounding planes is tangent to the sphere.

If a circle (Fig. 1) be revolved about a diameter PQ, the arc AD will generate a zone, the points A and D will generate its bases, and CF is its altitude. The arc PA will generate a zone of one base.

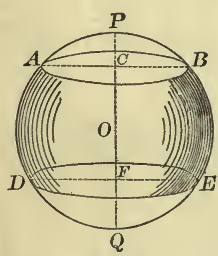


FIG. 1.

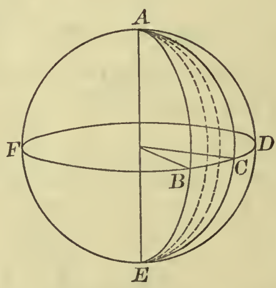


FIG. 2.

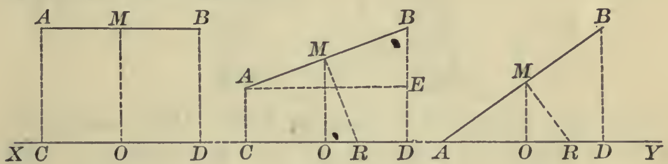
759. A *lune* is a portion of the surface of a sphere bounded by two semi-circumferences of great circles.

760. The *angle* of a lune is the angle between the semi-circumferences which form its boundaries. Thus (Fig. 2), $ABECA$ is a lune, BAC is its angle.

761. As in Plane Geometry it is convenient to divide a quadrant of a circle into 90 equal parts, called degrees, so in Solid Geometry it is convenient to divide each of the eight equal tri-rectangular triangles of which the surface of a sphere is composed (§ 744) into 90 equal parts, and to call these parts *spherical degrees.* The surface of every sphere therefore contains 720 spherical degrees.

Proposition XXVI. Theorem.

762. *The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis by the circumference whose radius is a perpendicular erected at the middle point of the line and terminated by the axis.*



Let XY be the axis, AB the revolving line, CD its projection on XY, M its middle point, MO perpendicular to XY, and MR perpendicular to AB.

To prove that area $AB = CD \times 2\pi MR$.

Proof. (1) If AB is $\parallel$ to XY, then $CD = AB$, MR coincides with MO, area AB is the surface of a right cylinder, and the truth of the theorem follows at once from § 646.

(2) If AB is not $\parallel$ to XY, area AB will be the surface of the frustum of a cone of revolution.

$\therefore$ area $AB = AB \times 2\pi MO$. § 676

Draw $AE \parallel$ to XY.

The $\triangle$ ABE, MOR are similar. § 327

$\therefore MO : AE = MR : AB$.

$\therefore AB \times MO = AE \times MR$.

Or, since $AE = CD$, § 180

$AB \times MO = CD \times MR$.

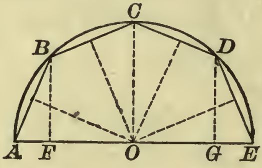
Substituting this value of $AB \times MO$ in the first equation, we obtain area $AB = CD \times 2\pi MR$.

(3) If A lies in the axis XY, the above reasoning still holds good; only AE and CD coincide, and the truth follows from § 670.

Q. E. D.

Proposition XXVII. Theorem.

763. *The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*



$2R$

$2\pi R$

$= 4\pi R^2$

Let the sphere be generated by the semicircle *ABCDE* revolving about the diameter *AE*, and let *O* be the centre, and *R* the radius.

To prove that the area of the surface $= AE \times 2\pi R$.

Proof. Inscribe in the semicircle half of a regular polygon having an *even* number of sides, as $ABCDE$.

From the centre draw ⊥s to the chords AB, BC, etc.

These ⊥s bisect the chords (§ 232) and are equal (§ 236).

Let a denote the length of each of these ⊥s.

From B and D drop the ⊥s BF and DG to AE.

When the semicircle revolves about AE, the sides of the polygon generate surfaces whose areas are as follows:

area $AB = AF \times 2\pi a$. § 762

area $BC = FO \times 2\pi a$.

area $CD = OG \times 2\pi a$.

area $DE = GE \times 2\pi a$.

Adding, area $ABCDE = AE \times 2\pi a$.

Now suppose the number of sides of the semi-polygon to be indefinitely increased; then the limit of the area $ABCDE$ is the area of the surface of the sphere, and the limit of a is R.

Hence the area of the surface of the sphere $= AE \times 2\pi R$. § 260

Q. E. D.

764. Cor. 1. If S denotes the area of the surface of a sphere, then by § 763,

$$S = 2R \times 2\pi R = 4\pi R^2.$$

But πR^2 is the area of a great circle; therefore,

The surface of a sphere is equivalent to four great circles.

765. Cor. 2. Let R and R' denote the radii, D and D' the diameters, and S and S' the areas of the surfaces of two spheres; then, by § 764,

$$S = 4\pi R^2, \quad S' = 4\pi R'^2.$$

$$\therefore \frac{S}{S'} = \frac{4\pi R^2}{4\pi R'^2} = \frac{R^2}{R'^2} = \frac{(\frac{1}{2}D)^2}{(\frac{1}{2}D')^2} = \frac{D^2}{D'^2}.$$

Therefore, *the areas of the surfaces of two spheres are as the squares of their radii, or as the squares of their diameters.*

766. Cor. 3. If we apply the reasoning of § 763 to the zone generated by the revolution of the arc BCD, we obtain for the result,

$$\text{area of zone } BCD = FG \times 2\pi R.$$

Now FG is the altitude of the zone; therefore,

The area of a zone is equal to the product of its altitude by the circumference of a great circle.

767. Cor. 4. *Zones on the same sphere, or equal spheres, are to each other as their altitudes.*

768. Cor. 5. The arc AB generates a zone of one base; and zone $AB = AF \times 2\pi R = \pi AF \times AE$. Now since $AF \times AE = AB^2$ (§ 337), the zone $AB = \pi AB^2$.

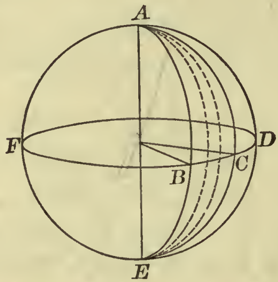
That is, *a zone of one base is equivalent to a circle whose radius is the chord of the generating arc.*

Ex. 583. Find the area of the surface of a sphere whose radius is 6 inches. 452 4/7

Ex. 584. Find the area of a zone if its altitude is 3 inches, and the radius of the sphere is 6 inches. = 22/7 113 1/7

Proposition XXVIII. Theorem.

769. *The area of a lune is to the area of the surface of the sphere as the number of degrees in its angle is to 360.*



Let ABEC be a lune, BCDF the great circle whose pole is A; also let A denote the number of degrees in the angle of the lune, L the area of the lune, and S the area of the surface of the sphere.

To prove that $L : S = A : 360.$

Proof. The arc BC measures the $\angle A$ of the lune. § 722

Hence, arc BC : circumference $BCDF = A : 360$.

(1) If BC and $BCDF$ are commensurable, let their common measure be contained m times in BC, and n times in $BCDF$. Then

$$\text{arc } BC : \text{circumference } BCDF = m : n.$$

$$\therefore A : 360 = m : n. \qquad \S\ 262$$

Pass arcs of great ⓢ through the diameter AE and all the points of the division of $BCDF$. These arcs will divide the entire surface into n equal lunes, of which the lune $ABEC$ will contain m.

$$\therefore L : S = m : n.$$

$$\therefore L : S = A : 360.$$

(2) If BC and $BCDF$ are incommensurable, the theorem can be proved by the method of limits as in § 261. Q. E. D.

770. Cor. 1. If L and S are expressed as spherical degrees (§ 761), then since S contains 720 spherical degrees,

$$L : 720 = A : 360°.$$

Whence $$L = 2A.$$

That is,

The numerical value of a lune expressed in spherical degrees is twice the numerical value of its angle expressed in angle-degrees.

771. Cor. 2. If L and S are expressed in ordinary units of area (as square inches, etc.), then, since $S = 4\pi R^2$,

$$L : 4\pi R^2 = A : 360°.$$

Whence $$L = \frac{\pi R^2 A}{90°}.$$

772. Cor. 3. If we compare two lunes on the same sphere, or equal spheres, R is constant; hence, if L, L' denote the lunes, A, A' their angles,

$$L : L' = \frac{\pi R^2 A}{90°} : \frac{\pi R^2 A'}{90°} = A : A'.$$

That is,

Two lunes on the same sphere, or equal spheres, have the same ratio as their angles.

773. Cor. 4. If we compare two lunes L, L', which have the same $\angle A$, but are situated on unequal spheres whose radii are R and R', then

$$L : L' = \frac{\pi R^2 A}{90°} : \frac{\pi R'^2 A}{90°} = R^2 : R'^2.$$

Two lunes on unequal spheres which have the same angle may be called *similar lunes*. Therefore,

Similar lunes have the same ratio as the squares of the radii of the spheres on which they are situated.

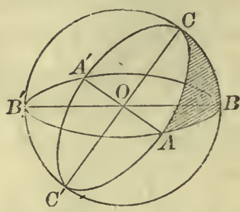
Ex. 585. Given the radius of a sphere 10 inches; find the area of a lune whose angle is 30°. $\therefore L = 104.7$

Ex. 586. Given the diameter of a sphere 16 inches; find the area of a lune whose angle is 75°.

$\therefore L = 167.47$

PROPOSITION XXIX. THEOREM.

774. *The area of a spherical triangle, expressed in spherical degrees, is numerically equal to the spherical excess of the triangle.*



Let A, B, C denote the values of the angles of the spherical triangle ABC, and E the spherical excess.

To prove that the number of spherical degrees in $\triangle\, ABC = E$.

Proof. Produce the sides of $\triangle\, ABC$ to complete circles.

These circles divide the surface of the sphere into eight spherical triangles, of which any four having a common vertex, as A, form the surface of a hemisphere, and therefore contain 360 spherical degrees.

$$\text{Now } \triangle\, ABC + \triangle\, A'BC \approx \text{lune } ABA'C.$$

And the $\triangle$s $A'BC$, $AB'C'$ are symmetrical.

$$\therefore \triangle\, A'BC \approx \triangle\, AB'C'. \qquad \S\ 748$$

$$\therefore \triangle\, ABC + \triangle\, AB'C' \approx \text{lune } ABA'C.$$

Also $\quad \triangle\, ABC + \triangle\, AB'C \approx \text{lune } BAB'C.$

And $\quad \triangle\, ABC + \triangle\, ABC' \approx \text{lune } CAC'B.$

Add and observe that in spherical degrees

$$\triangle\text{s } ABC + AB'C' + AB'C + ABC' = 360,$$

and lunes $ABA'C + BAB'C + CAC'B$ are numerically equal to $2(A + B + C)$, and we have § 770

$$2\, \triangle\, ABC + 360 = 2(A + B + C).$$

Whence $\quad \triangle\, ABC = A + B + C - 180 = E.$ Q. E. D.

775. Cor. 1. Since in spherical degrees $\triangle ABC = E$, and the entire surface of the sphere $= 720$, therefore,

$$\triangle ABC : \text{entire surface} = E : 720.$$

That is,

The area of a spherical triangle is to the area of the surface of the sphere as the number which expresses its spherical excess is to 720.

776. Cor. 2. Hence we may easily express the value of $\triangle ABC$ in ordinary units of area (as square inches, etc.).

For, let S denote the area of the surface of the sphere.

Then $$\triangle ABC : S = E : 720.$$

$$\therefore \triangle ABC = \frac{SE}{720}.$$

But $S = 4\pi R^2$ (§ 764).

$$\therefore \triangle ABC = \frac{4\pi R^2 E}{720} = \frac{\pi R^2 E}{180}.$$

Ex. 587. What part of the surface of a sphere is a triangle whose angles are 120°, 100°, and 95°? What is its area in square inches, if the radius of the sphere is 6 inches?

Ex. 588. Find the area of a spherical triangle whose angles are 100°, 120°, 140°, if the diameter of the sphere is 16 inches.

Ex. 589. If the radii of two spheres are 6 inches and 4 inches respectively, and the distance between their centres is 5 inches, what is the area of the circle of intersection of these spheres?

Ex. 590. Find the radius of the circle determined in a sphere of 5 inches diameter by a plane 1 inch from the centre.

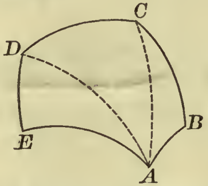
Ex. 591. If the radii of two concentric spheres are R and R', and if a plane is drawn tangent to the interior sphere, what is the area of the section made in the other sphere?

Ex. 592. Two points A and B are 8 inches apart. Find the locus in space of a point 6 inches from A and 7 inches from B.

Ex. 593. The radii of two parallel sections of the same sphere are a and b respectively, and the distance between these sections is d; find the radius of the sphere.

Proposition XXX. Theorem.

777. *If T denotes the sum of the angles of a spherical polygon of n sides, the area of the polygon expressed in spherical degrees is numerically equal to $T-(n-2)\,180°$.*



Let $ABCDE$ be a polygon of n sides.

To prove that the area of $ABCDE$ is numerically equal to $T-(n-2)\,180°$.

Proof. Divide the polygon into spherical triangles by drawing diagonals from any vertex, as A.

These diagonals will divide the polygon into $n-2$ spherical triangles, and the area of each triangle in spherical degrees is numerically equal to the sum of its angles minus 180°. § 774

Hence the sum of the areas of all the $n-2$ triangles is numerically equal to the sum of all their angles minus $(n-2)\,180°$.

Now the sum of the areas of the triangles is the area of the polygon, and the sum of their angles is the sum of the angles of the polygon, that is, T.

Therefore the area of the polygon is numerically equal to $T-(n-2)\,180°$.

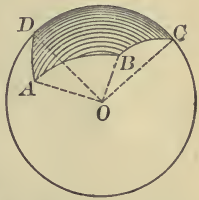
Ex. 594. Find the area of a spherical quadrangle whose angles are 170°, 139°, 126°, and 141°, if the radius of the sphere is 10 inches.

Ex. 595. Find the area of a spherical pentagon whose angles are 122°, 128°, 131°, 160°, 161°, if the surface of the sphere is 150 square feet.

Ex. 596. Find the area of a spherical hexagon whose angles are 96°, 110°, 128°, 136°, 140°, 150°, if the circumference of a great circle of the sphere is 10 inches.

THE VOLUME OF A SPHERE.

778. A *spherical pyramid* is the portion of a sphere bounded by a spherical polygon and the planes of its sides.



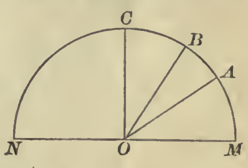
The centre of the sphere is the *vertex* of the pyramid.

The spherical polygon is its *base.*

Thus, $O\text{-}ABCD$ is a spherical pyramid.

779. A *spherical sector* is the portion of a sphere generated by the revolution of a circular sector about any diameter of the circle of which the sector is a part.

The *base* of a spherical sector is the zone generated by the arc of the circular sector. Thus, the circular sector AOB revolving about the line MN generates a spherical sector whose base is the zone generated by the arc AB; the other bounding surfaces are the conical surfaces generated by the radii OA and OB. The sector generated by AOM is bounded by a conical surface and a zone of one base If OC is perpendicular to OM, the sector generated by AOC is bounded by a conical surface, a plane surface, and a zone.



780. A *spherical segment* is a portion of a sphere contained between two parallel planes.

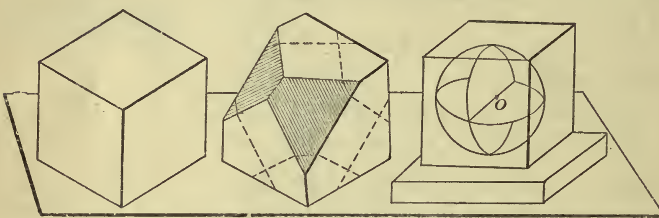
781. The *bases* of a spherical segment are the sections made by the parallel planes, and the *altitude* of a spherical segment is the distance between its bases.

782. If one of the parallel planes is tangent to the sphere, the segment is called a *segment of one base.*

783. A *spherical wedge* is a portion of a sphere bounded by a lune and two great semicircles.

Proposition XXXI. Theorem.

784. *The volume of a sphere is equal to the product of the area of its surface by one-third of its radius.*



Let R be the radius of a sphere whose centre is O, S its surface, and V its volume.

To prove $V = S \times \frac{1}{3} R$.

Proof. Conceive a cube to be circumscribed about the sphere. Its volume will be greater than that of the sphere, because it contains the sphere.

From O, the centre of the sphere, conceive lines to be drawn to the vertices of the cube.

These lines are the edges of six quadrangular pyramids, whose bases are the faces of the cube, and whose common altitude is the radius of the sphere.

The volume of each pyramid is equal to the product of its base by $\frac{1}{3}$ its altitude. Hence the volume of the six pyramids, that is, the volume of the circumscribed cube, is equal to the area of the surface of the cube multiplied by $\frac{1}{3} R$.

Now conceive planes drawn tangent to the sphere, at the points where the edges of the pyramids cut its surface. We shall then have a circumscribed solid whose volume will be nearer that of the sphere than is the volume of the circumscribed cube, because each tangent plane cuts away a portion of the cube.

From O conceive lines to be drawn to each of the polyhedral angles of the solid thus formed, a, b, c, etc.

These lines will form the edges of a series of pyramids, whose bases are the surface of the solid, and whose common altitude is the radius of the sphere; and the volume of each pyramid thus formed is equal to the product of its base by $\frac{1}{3}$ its altitude.

Hence the sum of the volumes of these pyramids, that is, the volume of this new solid, is again equal to the area of its surface multiplied by $\frac{1}{3}R$.

Now this process of drawing tangent planes may be considered as continued indefinitely, and, however far this process is carried, the volume of the solid will always be equal to the area of its surface multiplied by $\frac{1}{3}R$.

But the volume of the circumscribed solid will approach nearer and nearer to that of the sphere; and as the volumes approach coincidence, the surfaces also approach coincidence.

Hence, V and S are the limits of the volume and the surface respectively, of the circumscribed solid.

$$\therefore V = S \times \tfrac{1}{3}R. \qquad \S\, 260$$

Q. E. D.

785. Cor. 1. Since $S = 4\pi R^2$ (§ 764), and $R = \frac{1}{2}D$, we obtain by substitution the formulas

$$V = \tfrac{4}{3}\pi R^3, \text{ and } V = \tfrac{1}{6}\pi D^3.$$

786. Cor. 2. *The volumes of two spheres are to each other as the cubes of their radii.*

For, if R, R' denote the radii, V and V' the volumes,

$$V = \tfrac{4}{3}\pi R^3, \text{ and } V' = \tfrac{4}{3}\pi R'^3.$$

$$\therefore V : V' = \tfrac{4}{3}\pi R^3 : \tfrac{4}{3}\pi R'^3 = R^3 : R'^3.$$

787. Cor. 3. *The volume of a spherical pyramid is equal to the product of its base by one-third of the radius of the sphere.*

For, it is obvious that the reasoning employed in § 784 applies equally well to a spherical pyramid.

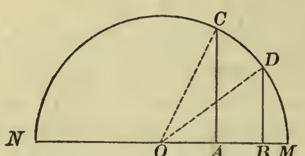
788. Cor. 4. *The volume of a spherical sector is equal to the product of the zone which forms its base by one-third of the radius of the sphere.*

789. Cor. 5. If R denotes the radius of a sphere, C the circumference of a great circle, H the altitude of the zone, Z the surface of the zone, and V the volume of the corresponding sector; then, since $C = 2\pi R$, and $Z = 2\pi RH$, we have

$$V = \tfrac{2}{3}\pi R^2 H.$$

Proposition XXXII. Problem.

790. *To find the volume of a spherical segment.*



Let AC and BD be two semi-chords perpendicular to the diameter MN of the semicircle $NCDM$. Let

$$OM = R,\ AM = a,\ BM = b,\ AB = a - b = h,\ AC = r,\ BD = r'.$$

Case I. *To find the volume of the segment of one base generated by the circular semi-segment ACM, as the semicircle revolves about NM as an axis.*

The sector generated by $OCM = \frac{2}{3}\pi R^2 a$. § 789

The cone generated by $OCA = \frac{1}{3}\pi r^2 (R - a)$. § 672

Hence segment $ACM = \frac{2}{3}\pi R^2 a - \frac{1}{3}\pi r^2 (R - a)$

$$= \frac{\pi}{3}(2R^2 a - Rr^2 + ar^2).$$

Now $r^2 = a(2R - a)$ (§ 337); therefore by substitution,

the segment $ACM = \pi a^2 \left(R - \dfrac{a}{3}\right)$. (1)

If from the relation $r^2 = a(2R - a)$ we find the value of R, and substitute it in (1), we obtain the volume in terms of the altitude and the radius of the base.

$$\text{The segment } ACM = \tfrac{1}{2}\pi r^2 a + \tfrac{1}{6}\pi a^3. \qquad (2)$$

CASE II. *To find the volume of the segment of two bases generated by the circular semi-segment $ABDC$, as the semicircle revolves about NM as an axis.*

Since the volume is obviously the difference of the volumes of the segments of one base generated by the circular semi-segments ACM and BDM, therefore by formula (1),

$$\text{segment } ABDC = \pi a^2\left(R - \frac{a}{3}\right) - \pi b^2\left(R - \frac{b}{3}\right)$$

$$= \pi R(a^2 - b^2) - \frac{\pi}{3}(a^3 - b^3) \qquad (3)$$

$$= \pi Rh(a + b) - \frac{\pi h}{3}(a^2 + ab + b^2)$$

$$= \pi h[(Ra + Rb) - \tfrac{1}{3}(a^2 + ab + b^2)].$$

Since $\quad a - b = h,\ a^2 - 2ab + b^2 = h^2;$

therefore $\quad a^2 + ab + b^2 = h^2 + 3ab;$

also since $\quad (2R - a)a = r^2$, and $(2R - b)b = r'^2$,

$$Ra + Rb = \frac{r^2 + r'^2}{2} + \frac{a^2 + b^2}{2}.$$

Hence

$$\text{the segment } ABDC = \pi h\left[\frac{r^2 + r'^2}{2} + \frac{a^2 + b^2}{2} - \frac{h^2}{3} - ab\right]$$

$$= \pi h\left[\frac{r^2 + r'^2}{2} + \frac{h^2}{2} + ab - \frac{h^2}{3} - ab\right]$$

$$= \frac{h}{2}(\pi r^2 + \pi r'^2) + \frac{\pi h^3}{6}.$$

$$= \frac{\pi h}{2}\left(r^2 + r_1^2 + \frac{h^2}{3}\right)$$

NUMERICAL EXERCISES.

597. Find the surface of a sphere if the diameter is (i.) 10 inches; (ii.) 1 foot 9 inches (iii.) 2 feet 4 inches; (iv.) 7 feet; (v.) 4.2 feet; (vi.) 10.5 feet.

598. Find the diameter of a sphere if the surface is (i.) 616 square inches; (ii.) $38\frac{1}{2}$ square feet; (iii.) 9856 square feet.

599. The circumference of a dome in the shape of a hemisphere is 66 feet; how many square feet of lead are required to cover it?

600. If the ball on the top of St. Paul's Cathedral in London is 6 feet in diameter, what would it cost to gild it at 7 cents per square inch?

601. What is the numerical value of the radius of a sphere if its surface has the same numerical value as the circumference of a great circle?

602. Find the surface of a lune if its angle is 30°, and the total surface of the sphere is 4 square feet.

603. What fractional part of the whole surface of a sphere is a spherical triangle whose angles are 43° 27′, 81° 57′, and 114° 36′?

604. The angles of a spherical triangle are 60°, 70°, and 80°. The radius of the sphere is 14 feet. Find the area of the triangle in square feet.

605. The sides of a spherical triangle are 38°, 74°, and 128°. The radius of the sphere is 14 feet. Find the area of the polar triangle in square feet.

606. Find the area of a spherical polygon on a sphere whose radius is $10\frac{1}{2}$ feet, if its angles are 100°, 120°, 140°, and 160°.

607. The planes of the faces of a quadrangular spherical pyramid make with each other angles of 80°, 100°, 120°, and 150°; and the length of a lateral edge of the pyramid is 42 feet. Find the area of its base in square feet.

608. The planes of the faces of a triangular spherical pyramid make with each other angles of 40°, 60°, and 100°, and the area of the base of the pyramid is 4π square feet. Find the radius of the sphere.

609. The diameter of a sphere is 21 feet. Find the curved surface of a segment whose height is 5 feet.

610. What is the area of a zone of one base whose height is h, and the radius of the base r? What would be the area if the height were twice as great?

611. In a sphere whose radius is r, find the height of a zone whose area is equal to that of a great circle.

612. The altitude of the torrid zone is about 3200 miles. Find its area in square miles, assuming the earth to be a sphere with a radius of 4000 miles.

613. A plane divides the surface of a sphere of radius r into two zones, such that the surface of the greater is a mean proportional between the entire surface and the surface of the smaller. Find the distance of the plane from the centre of the sphere.

614. If a sphere of radius r is cut by two planes equally distant from the centre, so that the area of the zone comprised between the planes is equal to the sum of the areas of its bases, find the distance of either plane from the centre.

615. Find the area of the zone generated by an arc of 30°, of which the radius is r, and which turns around a diameter passing through one of its extremities.

616. Find the area of the zone of a sphere of radius r, illuminated by a lamp placed at the distance h from the sphere.

617. How much of the earth's surface would a man see if he were raised to the height of the radius above it?

618. To what height must a man be raised above the earth in order that he may see one-sixth of its surface?

619. Two cities are 200 miles apart. To what height must a man ascend from one city in order that he may see the other, supposing the circumference of the earth to be 25,000 miles?

620. Find the volume of a sphere if the diameter is (i.) 13 inches; (ii.) 3 feet 6 inches; (iii.) 10 feet 6 inches; (iv.) 17 feet 6 inches; (v.) 14.7 feet; (vi.) 42 feet.

621. Find the diameter of a sphere if the volume is (i.) 75 cubic feet 1377 cubic inches; (ii.) 179 cubic feet 1152 cubic inches; (iii.) 1047.816 cubic feet; (iv.) 38.808 cubic yards.

622. Find the volume of a sphere whose circumference is 45 feet.

623. Find the volume V of a sphere in terms of the circumference C of a great circle.

624. Find the radius r of a sphere, having given the volume V.

625. Find the radius r of a sphere, if its circumference and its volume have the same numerical value.

626. If an iron ball 4 inches in diameter weighs 9 pounds, what is the weight of a hollow iron shell 2 inches thick, whose external diameter is 20 inches?

627. The radius of a sphere is 7 feet; what is the volume of a wedge whose angle is 36°?

628. What is the angle of a spherical wedge, if its volume is one cubic foot, and the volume of the entire sphere is 6 cubic feet?

629. What is the volume of a spherical sector, if the area of the zone which forms its base is 3 square feet, and the radius of the sphere is 1 foot?

630. The radius of the base of the segment of a sphere is 16 inches, and the radius of the sphere is 20 inches; find its volume.

631. The inside of a wash-basin is in the shape of the segment of a sphere; the distance across the top is 16 inches, and its greatest depth is 6 inches; find how many pints of water it will hold, reckoning $7\frac{1}{2}$ gallons to the cubic foot.

632. What is the height of a zone, if its area is S, and the volume of the sphere to which it belongs is V?

633. The radii of the bases of a spherical segment are 6 feet and 8 feet, and its height is 3 feet; find its volume.

634. Find the volume of a triangular spherical pyramid if the angles of the spherical triangle which forms its base are each 100°, and the radius of the sphere is 7 feet.

635. The circumference of a sphere is 28π feet; find the volume of that part of the sphere included by the faces of a trihedral angle at the centre, the dihedral angles of which are 80°, 105°, and 140°.

636. The planes of the faces of a quadrangular spherical pyramid make with each other angles of 80°, 100°, 120°, and 150°, and a lateral edge of the pyramid is $3\frac{1}{2}$ feet; find the volume of the pyramid.

637. The radius of the base of the segment of a sphere is 40 feet, and its height is 20 feet; find its volume.

638. Having given the volume V, and the height h, of a spherical segment of one base, find the radius r of the sphere.

639. Find the weight of a sphere of radius r, which floats in a liquid of specific gravity s, with one-fourth of its surface above the surface of the liquid. *The weight of a floating body is equal to the weight of the liquid displaced.*

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Miscellaneous Exercises.

640. Determine a point in a given plane such that the difference of its distances from two given points on opposite sides of the plane shall be a maximum.

641. In any warped quadrilateral, that is, one whose sides do not all lie in the same plane, the middle points of the sides are the vertices of a parallelogram.

642. In any trihedral angle, the three planes bisecting the three dihedral angles intersect in the same straight line.

643. To draw a line through the vertex of any trihedral angle, making equal angles with its edges.

644. In any trihedral angle, the three planes passed through the edges and the respective bisectors of the opposite face angles intersect in the same straight line.

645. In any trihedral angle, the three planes passed through the bisectors of the face angles, and perpendicular to these faces respectively, intersect in the same straight line.

646. In any trihedral angle, the three planes passed through the edges, perpendicular to the opposite faces respectively, intersect in the same straight line.

647. In a tetrahedron, the planes passed through the three lateral edges and the middle points of the sides of the base intersect in a straight line.

648. The lines joining each vertex of a tetrahedron with the point of intersection of the medial lines of the opposite face all meet in a point called the *centre of gravity*, which divides each line so that the shorter segment is to the whole line in the ratio 1 : 4.

649. The straight lines joining the middle points of the opposite edges of a tetrahedron all pass through the centre of gravity of the tetrahedron, and are bisected by the centre of gravity.

650. The plane which bisects a dihedral angle of a tetrahedron divides the opposite edges into segments proportional to the areas of the faces including the dihedral angle.

651. The altitude of a regular tetrahedron is equal to the sum of the four perpendiculars let fall from any point within it upon the four faces.

652. Within a given tetrahedron, to find a point such that the planes passed through this point and the edges of the tetrahedron shall divide the tetrahedron into four equivalent tetrahedrons.

653. To cut a cube by a plane so that the section shall be a regular hexagon.

654. To cut a tetrahedral angle so that the section shall be a parallelogram.

655. The portion of a tetrahedron cut off by a plane parallel to any face is a tetrahedron similar to the given tetrahedron.

656. Two tetrahedrons, having a dihedral angle of one equal to a dihedral angle of the other, and the faces including these angles respectively similar, and similarly placed, are similar.

657. Two polyhedrons composed of the same number of tetrahedrons, similar each to each and similarly placed, are similar.

658. If the homologous faces of two similar pyramids are respectively parallel, the straight lines which join the homologous vertices of the pyramids meet in a point.

659. Two symmetrical tetrahedrons are equivalent.

660. Two symmetrical polyhedrons may be decomposed into the same number of tetrahedrons symmetrical each to each.

661. Two symmetrical polyhedrons are equivalent.

662. If a solid has two planes of symmetry perpendicular to each other, the intersection of these planes is an axis of symmetry of the solid.

663. If a solid has three planes of symmetry perpendicular to each other, the three intersections of these planes are three axes of symmetry of the solid; and the common intersection of these axes is the centre of symmetry of the solid.

664. The volume of a right circular cylinder is equal to the product of the lateral area by half the radius.

665. The volume of a right circular cylinder is equal to the product of the area of the rectangle which generates it, by the length of the circumference generated by the point of intersection of the diagonals of the rectangle.

666. If the altitude of a right circular cylinder is equal to the diameter of the base, the volume is equal to the total area multiplied by a third of the radius.

Construct a spherical surface with given radius:

667. Passing through three given points.

668. Passing through two given points and tangent to a given plane.

669. Passing through two given points and tangent to a given sphere.

670. Passing through a given point and tangent to two given planes.

671. Passing through a given point and tangent to two given spheres.

672. Passing through a given point and tangent to a given plane and a given sphere.

673. Tangent to three given planes.

674. Tangent to three given spheres.

675. Tangent to two given planes and a given sphere.

676. Tangent to two given spheres and a given plane.

677. Find the area of a solid generated by an equilateral triangle turning about one of its sides, if the length of the side is a.

678. Find the centre of a sphere whose surface shall pass through three given points, and shall touch a given plane.

679. Find the centre of a sphere whose surface shall touch two given planes, and also pass through two given points which lie between the planes.

680. Through a given point to pass a plane tangent to a given circular cylinder.

681. Through a given point to pass a plane tangent to a given circular cone.

682. Through a given straight line without a given sphere, to pass a plane tangent to the sphere.

683. The volume of a sphere is two-thirds of the volume of a circumscribing cylinder, and its surface is two-thirds of the total surface of the cylinder.

684. Given a sphere, a cylinder circumscribed about the sphere, and a cone of two nappes inscribed in the cylinder; if any two planes are drawn perpendicular to the axis of the three figures, the spherical segment between the planes is equivalent to the difference between the corresponding cylindrical and conic segments.

685. Compare the volumes of the solids generated by a rectangle turning successively about two adjacent sides, the lengths of these sides being a and b.

686. An equilateral triangle revolves about one of its altitudes. Compare the convex surface of the cone generated by the triangle and the surface of the sphere generated by the circle inscribed in the triangle.

687. An equilateral triangle revolves about one of its altitudes. Compare the volumes of the solids generated by the triangle, the inscribed circle, and the circumscribed circle.

688. The perpendicular let fall from the point of intersection of the medial lines of a given triangle upon any plane not cutting the triangle is equal to one-third the sum of the perpendiculars from the vertices of the triangle upon the same plane.

689. The perpendicular from the centre of gravity of a tetrahedron upon any plane not cutting the tetrahedron is equal to one-fourth the sum of the perpendiculars from the vertices of the tetrahedron upon the same plane.

690. The volume of any polyhedron having for its bases any two polygons whose planes are parallel, and for lateral faces trapezoids, is the product of one-sixth the distance between the bases into the sum of the two bases plus four times a section midway between the bases; that is, if H denotes the distance between the bases B and b, and B' a section midway between the bases,

$$V = \tfrac{1}{6} H(B + b + 4B').$$

NOTE. From any point O in the section midway between the bases, draw lines to the vertices of the solid angles of the polyhedron, thus dividing the solid into pyramids. The pyramids having B and b as bases, evidently equal $\frac{1}{6} H(B + b)$. It remains to be proved that the volume of each pyramid having a lateral face as its base equals $\frac{1}{6} H$ into four times that portion of the section midway between the bases intercepted by this pyramid. This theorem is much used in earth-work.

BOOK IX.

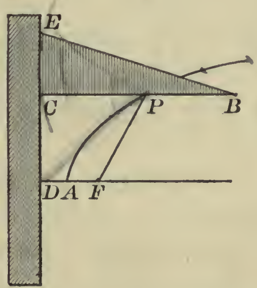
CONIC SECTIONS.

THE PARABOLA.

791. The curve traced by a point which moves so that its distance from a fixed point is always equal to its distance from a fixed line is called a *parabola*. The curve lies in the plane of the fixed point and line.

792. The fixed point is called the *focus;* and the fixed line, the *directrix*.

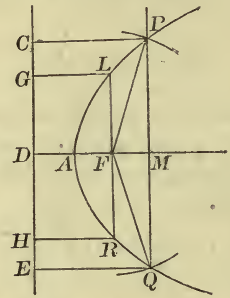
793. A parabola may be described by the continuous motion of a point, as follows:



Place a ruler so that one of its edges shall coincide with the directrix DE. Then place a right triangle with its base edge in contact with the edge of the ruler. Fasten one end of a string, whose length is equal to the other edge BC, to the point B, and the other end to a pin fixed at the focus F. Then slide the triangle BCE along the directrix, keeping the string tightly pressed against the ruler by the point of a pencil P. The point P will describe a parabola; for during the motion we always have $PF = PC$.

Proposition I. Problem.

794\. *To construct a parabola by points, having given its focus and its directrix.*



Let F be the focus, and CDE the directrix.

To construct the parabola by points.

Construction. Draw $FD \perp$ to CE, and meeting CE at D.

Bisect FD at A. Then A is a point of the curve. § 791

Through any point M in the line DF, to the right of A, draw a line $\parallel$ to CE.

With F as centre and DM as radius, draw arcs cutting this line at the points P and Q.

Then P and Q are points of the curve.

Proof. Draw PC, $QE \perp$ to CE.

Then $PC = DM$, and $QE = DM$,

and $DM = PF = QF$. Cons.

$\therefore PC = PF$; and $QE = QF$.

Therefore P and Q are in the curve. § 791

In this way any number of points may be found; and a continuous curve drawn through the points thus determined will be the parabola whose focus is F and directrix CDE.

Q. E. F.

795. The point A is called the *vertex* of the curve. The line DF produced indefinitely in both directions is called the *axis.*

796. The line FP, joining the focus to any point P on the curve, is called the *focal radius* of P.

797. The distance AM is called the *abscissa*, and the distance PM the *ordinate*, of the point P.

798. The double ordinate LR, through the focus, is called the *latus rectum* or *parameter.*

799. Cor. 1. Since $FP = FQ$ (Cons.), $MP = MQ$ (§ 121); hence, *the parabola is symmetrical with respect to its axis* (§ 63).

800. Cor. 2. *The curve lies entirely on one side of the perpendicular to the axis erected at the vertex; namely, on the same side as the focus.*

For, any point on the other side of this perpendicular is obviously nearer to the directrix than to the focus.

801. Cor. 3. *The parabola is not a closed curve.*

For any point on the axis of the curve to the right of F is evidently nearer to the focus than to the directrix. Hence the parabola QAP cannot cross the axis to the right of F.

802. Cor. 4. *The latus rectum is equal to* $4\,AF$.

For, draw $LG \perp$ to CDE.

Then, $LF = LG$, and $LG = DF$.

$$\therefore LF = DF = 2\,AF.$$

Similarly, $RF = DF = 2\,AF$.

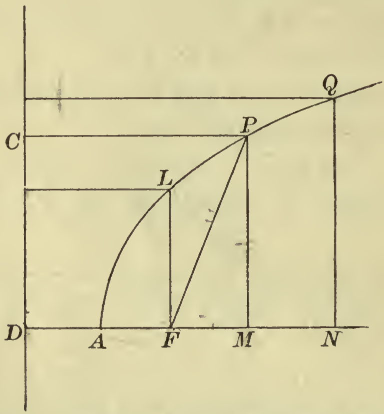
Therefore $LR = 4\,AF$.

---

803. Remark. In the following propositions, the focus will be denoted by F, the vertex by A, and the point where the axis meets the directrix by D.

PROPOSITION II. THEOREM.

804. *The ordinate of any point of a parabola is a mean proportional between the latus rectum and the abscissa.*



Let P be any point, AM its abscissa, PM its ordinate.

To prove $\overline{PM}^2 = 4\,AF \times AM.$

Proof. $\overline{PM}^2 = \overline{FP}^2 - \overline{FM}^2 = \overline{DM}^2 - \overline{FM}^2$ § 791

$$= (DM - FM)(DM + FM)$$
$$= DF(DF + FM + FM)$$
$$= 2\,AF(2\,AF + 2\,FM)$$
$$= 2\,AF(2\,AM).$$

Hence $\overline{PM}^2 = 4\,AF \times AM.$ (1) Q. E. D.

805. COR. 1. *The greater the abscissa of a point, the greater the ordinate.* For PM increases with AM in equation (1).

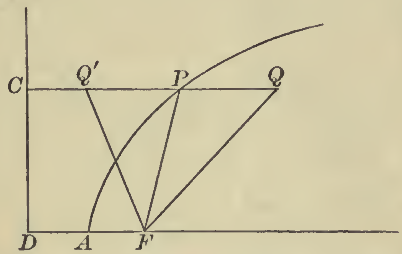
806. COR. 2. If P and Q are any two points of the curve,

$$\frac{\overline{PM}^2}{\overline{QN}^2} = \frac{4\,AF \times AM}{4\,AF \times AN} = \frac{AM}{AN}.$$

Hence, *the squares of any two ordinates are as the abscissas.*

Proposition III. Theorem.

807. *Every point within the parabola is nearer to the focus than to the directrix; and every point without the parabola is farther from the focus than from the directrix.*



1. Let Q be a point within the parabola. Draw QC perpendicular to the directrix, cutting the curve at P. Draw QF, PF.

To prove $QF < QC$.

Proof. In the $\triangle QPF$, $QF < QP + PF$. § 137

But $PF = PC$. § 791

$\therefore QF < QP + PC$,

$\therefore QF < QC$.

2. Let Q' be a point without the curve. Draw $Q'F$.

To prove $Q'F > Q'C$.

Proof. In the $\triangle Q'FP$, $Q'F > PF - PQ'$, § 137

or $Q'F > PC - PQ'$.

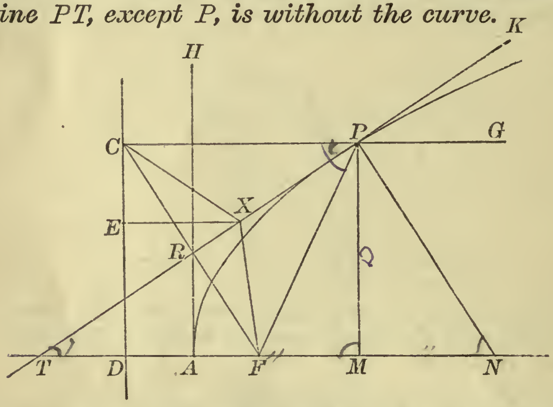
That is, $Q'F > Q'C$. Q. E. D.

808. Cor. *A point is within or without a parabola according as its distance from the focus is less than, or greater than, its distance from the directrix.*

809. A straight line which touches, but does not cut, a parabola, is called a *tangent* to the parabola. The point where it touches the parabola is called the *point of contact.*

Proposition IV. Theorem.

810. *If a line PT is drawn from any point P of the curve, bisecting the angle between PF and the perpendicular from P to the directrix, every point of the line PT, except P, is without the curve.*



Let PC be the perpendicular from P to the directrix, the angle FPT equal the angle CPT, and let X be any other point in PT except P.

To prove that X is without the curve.

Proof. Draw $XE \perp$ to the directrix, and join CX, FX, CF, and let CF meet PT at R.

In the isos. $\triangle PCF$,	$CR = RF$.	Ex. 14
Hence	$CX = FX$.	§ 122
But	$EX < CX$.	§ 114
Therefore	$EX < FX$.	
That is,	X is without the curve.	§ 808

Q. E. D.

811. Cor. 1. *PT is the tangent at the point P* (§ 809).

812. Cor. 2. *PT bisects FC, and is perpendicular to FC.*

813. Cor. 3. *Since the angles FPT and FTP are equal, FT equals FP* (§ 156).

814. Cor. 4. *The tangent at A is perpendicular to the axis.*
For it bisects the straight angle FAD.

315. Cor. 5. *The tangent at A is the locus of the foot of the perpendicular dropped from the focus to any tangent.*
Since $FR = RC$, and $FA = AD$, R is in AH (§ 311).

816. The line PN drawn through P perpendicular to the tangent PT is called the *normal* at P.

817. If the ordinate of P meet the axis in M, and the tangent and normal at P meet the axis in T and N respectively, then MT is the *subtangent* and MN the *subnormal.*

✓ 818. Cor. 6. *The subtangent is bisected by the vertex.*

For, $FT = FP$, § 813

and $FP = DM$. § 791

Hence $FT = DM$; also $AF = AD$.

Therefore $FT - AF = DM - AD$,

or $TA = AM$.

✓ 819. Cor. 7. *The subnormal is equal to half the latus rectum.*

For $CP = FN$, and $CP = DM$. § 180

Hence $FN = DM$,

or $FM + MN = DF + FM$.

Therefore $MN = DF$.

820. Cor. 8. *The normal bisects the angle between FP and CP produced; that is, bisects the angle FPG.*

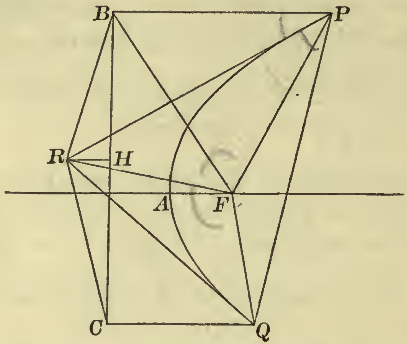
For $\angle NPT = \angle NPK$, and $\angle FPT = \angle TPC = \angle GPK$.

Hence $\angle NPF = \angle NPG$.

821. Cor. 9. *The circle with F as centre and FP as radius passes through T and N.*

Proposition V. Problem.

822. *To draw a tangent to a parabola from an exterior point.*



Let R ***be any point exterior to the parabola*** QAP.

To draw a tangent from R *to* QAP.

Construction. With R as centre and RF as radius, draw arcs cutting the directrix at the points B, C. Through B and C draw lines parallel to the axis, and meeting the parabola in P, Q, respectively. Join RP, RQ. Then RP and RQ are tangents to the curve.

Proof.	$RB = RF$,	Cons.
	$PB = PF$.	§ 791
Hence	$\angle RPB = \angle RPF$.	§ 160
Therefore	RP is the tangent at P.	§ 811
For like reason, RQ is the tangent at Q.		Q. E. F.

823. Cor. *Since* R *is without the curve, it is nearer to the directrix than to the focus* (§ 807); *therefore, the circle with* R *as centre and* RF *as radius, must always cut the directrix in two points; therefore, two tangents can always be drawn to a parabola from an exterior point.*

824. The line joining the points of contact P and Q is called the *chord of contact* for the tangents drawn from R.

Proposition VI. Theorem.

825. *The line joining the focus to the intersection of two tangents makes equal angles with the focal radii drawn to the points of contact.*

Let the tangents drawn at P and Q meet in R.

To prove $\angle RFP = \angle RFQ$.

Proof. Draw the ⊥s PB, QC to the directrix, and join RB, RC, RF.

Since $PB = PF$,

and $RB = RF$, §§ 812, 112

$\triangle RFP = \triangle RBP$, § 160

and $\angle RFP = \angle RBP$.

Similarly, $\angle RFQ = \angle RCQ$.

Now, $\angle RBP = 90° + \angle RBC$,

and $\angle RCQ = 90° + \angle RCB$;

and since $RB = RF$, and $RC = RF$,

therefore $RB = RC$.

Hence $\angle RBC = \angle RCB$. § 154

Therefore $\angle RBP = \angle RCQ$,

and $\angle RFP = \angle RFQ$. Q. E. D.

826. Cor. If the chord of contact PQ passes through F, then PFQ is a straight line.

Hence $RFP + RFQ = 180°$,

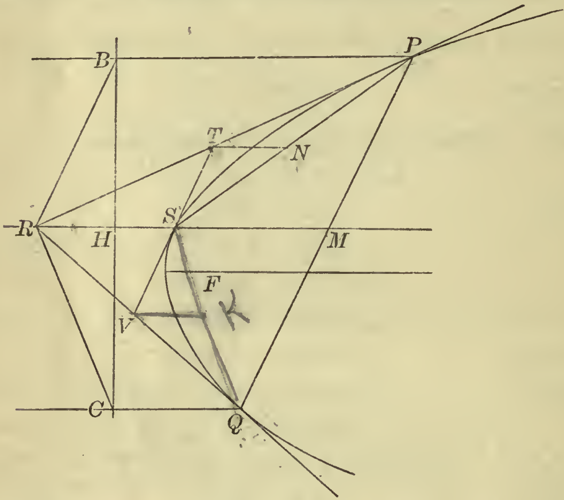
and $RFP = RFQ = 90°$.

Therefore $RBP = RCQ = 90°$.

Therefore, *the tangents drawn through the ends of a focal chord meet in the directrix.*

Proposition VII. Theorem.

827. *If a pair of tangents are drawn from a point R to a parabola, the line drawn through R parallel to the axis will bisect the chord of contact.*



Let the tangents drawn from R meet the curve in P, Q, and let the line through R parallel to the axis meet the directrix in H, the curve in S, and the chord of contact in M.

To prove $PM = QM$.

Proof. Drop the ⊥s PB, QC to the directrix,

and join RB, RC.

RH is ⊥ to BC, § 102

$RB = RC$. § 823

Hence $BH = CH$. § 121

Since PB, QC, and RM are ∥, § 100

therefore $PM = QM$. § 187

Q. E. D

PROPOSITION VIII. THEOREM.

828. *If a pair of tangents RP, RQ are drawn from a point R to a parabola, and through R a line parallel to the axis is drawn, meeting the curve in S, the tangent at S will be parallel to the chord of contact.*

Let the tangent at S meet the tangents PR, QR in T, V, respectively.

To prove $TV \parallel$ to PQ.

Proof. Draw $TN \parallel$ to SM, and let it meet SP in N.

Then	$PN = NS$.	§ 827
Hence	$PT = TR$.	§ 188
Similarly,	$QV = VR$.	
Therefore	TV is $\parallel$ to PQ.	§ 189

Q. E. D.

✓ 829. COR. 1. If we suppose R to move along RM towards the curve, then since the point S and the direction of the tangent TV remain fixed, the chord PQ will remain parallel to TV, while its middle point M will move along RM towards S; finally, R, M, P, and Q will all coincide at S.

Hence, *the line RM is the locus of the middle points of all chords drawn parallel to the tangent at S.*

830. The locus of the middle points of a system of parallel chords in a parabola is called a *diameter*. The parallel chords are called the *ordinates* of the diameter.

831. COR. 2. *The diameters of a parabola are parallel to its axis; and conversely, every straight line parallel to the axis is a diameter; that is, bisects a system of parallel chords.*

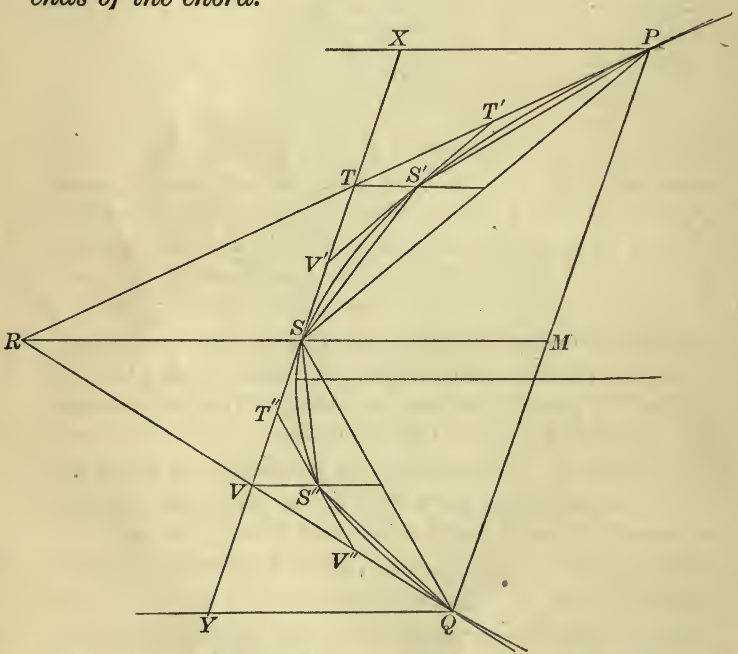
832. COR. 3. *Tangents drawn through the ends of an ordinate intersect in the diameter corresponding to that ordinate.*

833. COR. 4. The point S is the middle point of RM (§ 188); therefore, *the portion of a diameter contained between any ordinate and the intersection of the tangents drawn through the ends of the ordinate is bisected by the curve.*

834. Cor. 5. The point S is also the middle point of the tangent TV; therefore, *the part of a tangent parallel to a chord contained between the two tangents drawn through the ends of the chord is bisected by the diameter of the chord at the point of contact.*

Proposition IX. Theorem.

835. *The area of a parabolic segment made by a chord is two-thirds the area of the triangle formed by the chord and the tangents drawn through the ends of the chord.*



Let PQ be any chord, and let the tangents at P and Q meet in R.

To prove segment $PSQ = \frac{2}{3} \triangle PRQ$.

Proof. Draw the diameter RM, meeting the curve at S, and at S draw a tangent meeting PR in T and QR in V. Join SP, SQ.

Since $PT = TR$, and $QV = VR$, § 828

VT is ‖ to PQ,

and $PQ = 2 \times VT$. § 189

$\therefore \triangle PQS = 2 \triangle TVR$. § 370

If now we draw through T, V, the diameters TS', VS'', and then draw through S', S'', the tangents $T'S'V'$, $T''S''V''$, we can prove in the same way that

$$\triangle PSS' = 2 \triangle T'V'T,$$

and

$$\triangle QSS'' = 2 \triangle T''V''V.$$

If we continue to form new triangles by drawing diameters through the points T', V', T'', V'', and tangents at the points where these diameters meet the curve, we can prove that each interior triangle formed by joining a point of contact to the extremities of a chord is twice as large as the exterior triangle formed by the tangents through these points. And this is true however long the process is continued.

Therefore the sum of all the interior triangles is equal to twice the sum of the corresponding exterior triangles.

Now if we suppose the process to be continued indefinitely, then the limit of the sum of the interior triangles will be the area contained between the chord PQ and the curve, and the limit of the sum of the exterior triangles will be the area contained between the curve and the tangents PR, QR.

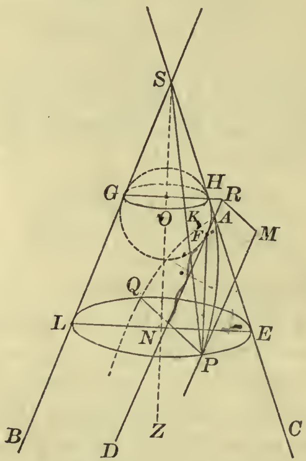
Hence segment $PQS =$ twice the area contained by PR, QR, and the curve, $= \frac{2}{3} \triangle PQR$. § 260

Q. E. D.

836. Cor. *If through P and Q lines are drawn parallel to SM, meeting the tangent TV produced in the points X and Y, then the segment $PQS = \frac{2}{3} \square\, PQYX$.*

Proposition X. Theorem.

837. *The section of a right circular cone made by a plane parallel to one, and only one, element of the surface is a parabola.*



Let SB be any element of the cone whose axis is SZ, and let QAP be the section of the cone made by a plane perpendicular to the plane BSZ and parallel to SB.

To prove that the curve PAQ is a parabola.

Proof. Let SC be the second element in which the plane BSZ cuts the cone, and let RAD be the intersection of the planes BSZ and PAQ.

Draw the ⊙ O tangent to the lines SB, SC, RD, and let G, H, F be the points of contact respectively.

Revolve BSC and the ⊙ OGH about the axis SZ, the plane PAQ remaining fixed. The ⊙ O will generate a sphere which will touch the cone in the ⊙ GKH, and the plane PAQ at the point F.

Since SO is $\perp$ to GH, SO is $\perp$ to the plane GKH. § 462

Hence the plane BSC is $\perp$ to the plane GKH. § 518

Let the plane of the $\odot$ GKH intersect the plane of the curve PAQ in the straight line MR; then will MR be $\perp$ to the plane BSC (§ 520), and therefore $\perp$ to DR.

Take any point P in the curve, and draw SP meeting the $\odot$ GH in K; join FP, and draw $PM \perp$ to RM.

Pass a plane through $P \perp$ to the axis of the cone. Let it cut the cone in the $\odot$ $EPLQ$, and the plane of the curve PAQ in the line PNQ.

PN is $\perp$ to the plane BSC (§ 520), and therefore $\perp$ to DR.

Since PF and PK are tangents to the sphere O, they are tangents to the circle of the sphere made by a plane passing through the points P, F, K, and are therefore equal. § 246

That is, $$PF = PK.$$

But $$PK = LG. \qquad \text{§ 666}$$

$$\therefore PF = LG. \qquad (1)$$

Now LG and PM are each $\parallel$ to NR;

hence LG is $\parallel$ to PM. § 485

The planes GKH and LPE are parallel. § 491

$$\therefore LG = PM. \qquad \text{§ 493}$$

From (1) and the last equation, we have

$$PF = PM.$$

That is, any point P on the curve PAQ is equidistant from a fixed point F and a fixed line RM in its plane.

Therefore the curve PAQ is a parabola.

Q. E. D

Exercises.

691. Prove that if the abscissa of a point is equal to its ordinate, each is equal to the latus rectum.

692. To draw a tangent and a normal at a given point of a parabola.

693. To draw a tangent to a parabola parallel to a given line.

694. Show that the tangents at the ends of the latus rectum meet at D.

695. Prove that the latus rectum is the shortest focal chord.

696. The tangent at any point meets the directrix and the latus rectum produced at points equally distant from the focus.

697. The circle whose diameter is FP touches the tangent at A.

698. The directrix touches the circle having any focal chord for diameter.

699. Given two points and the directrix, to find the focus.

700. The $\perp FC$ bisects TP. (See figure, page 392.)

701. Given the focus and the axis, to describe a parabola which shall touch a given straight line.

702. If PN is any normal, and $\triangle PNF$ is equilateral, then PF is equal to the latus rectum.

703. Given a parabola, to find the directrix, axis, and focus.

704. To find the locus of the centre of a circle which passes through a given point and touches a given straight line.

705. Given the axis, a tangent, and the point of contact, to find the focus and directrix.

706. Given two points and the focus, to find the directrix.

The Ellipse.

838. The locus of a point which moves so that the sum of its distances from two fixed points is constant is called an *ellipse.*

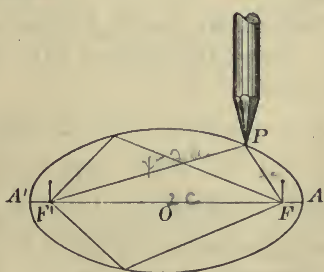
The fixed points are called the *foci*, and the straight lines which join a point of the curve to the foci are called *focal radii.*

The constant sum is denoted by $2a$, and the distance between the foci by $2c$.

The ratio $c:a$ is called the *eccentricity*, and is denoted by e. $\frac{c}{a}$
Therefore $c = ae$. $c =$

839. Cor. $2a$ *must be greater than* $2c$ (§ 137); *hence* e *must be less than* 1.

840. The curve may be described by the continuous motion of a point, as follows:

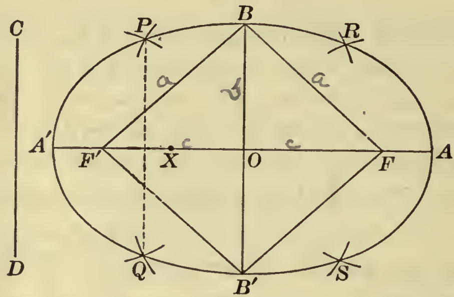


Fasten the ends of a string, whose length is $2a$, at the foci F and F'. Trace a curve with the point P of a pencil pressed against the string so as to keep it stretched. The curve thus traced will be an ellipse whose foci are F and F', and in which the constant sum of the focal radii is $FP + PF'$.

The curve is a closed curve extending around both foci; if it cuts FF' produced in A and A', it is easy to see that AA' equals the length of the string.

Proposition XI. Problem.

841. *To construct an ellipse by points, having given the foci and the constant sum* $2a$.



Let F and F' be the foci, and $2\,CD = 2a$.

To construct the ellipse.

Construction. Through the foci F, F' draw a straight line; bisect FF' at O. Lay off $OA' = OA = CD$. Then A, A' are two points of the curve.

Proof. From the construction, $AA' = 2a$, and $AF = A'F'$.

Therefore $AF + AF' = A'F + A'F' = AA' = 2a$,

and $A'F + A'F' = A'F + AF = AA' = 2a$.

To locate other points, mark any point X between F and F'. Describe an arc with F as centre and AX as radius; also another arc with F' as centre and $A'X$ as radius; let these arcs cut in P, Q.

Then P, Q are two points of the curve.

This follows at once from the construction and § 838.

By describing the same arcs *with the foci interchanged*, two more points R, S may be found.

By assuming other points between F and F', and proceeding in the same way, any number of points may be found.

The curve passing through all the points is an ellipse having F, F' for foci, and $2a$ for the constant sum of focal radii.
Q. E. F.

842. Cor. 1. By describing arcs from the foci with the same radius OA, we obtain two points B, B' of the curve such that they are *equidistant from the foci.* Therefore the *line BB' is perpendicular to AA' and passes through O* (§ 123).

843. The point O is called the *centre.* The line AA' is called the *major axis;* its ends A, A' are called the *vertices* of the curve. The line BB' is called the *minor axis.* The length of the minor axis is denoted by $2b$.

844. Cor. 2. *The major axis is bisected at O, and is equal to the constant sum 2a.*

845. Cor. 3. *The minor axis is also bisected at O* (§ 123).

Therefore $$OB = OB' = b.$$

846. Cor. 4. *The values of a, b, c are so related that*

$$a^2 = b^2 + c^2.$$

For, in the rt. $\triangle$ BOF,

$$\overline{BF}^2 = \overline{OB}^2 + \overline{OF}^2.$$

847. Cor. 5. The axis AA' bisects PQ at right angles (§ 123). Hence *the ellipse is symmetrical with respect to its major axis.*

848. The distance of a point of the curve from the minor axis is called the *abscissa* of the point, and its distance from the major axis is called the *ordinate* of the point.

The double ordinate through the focus is called the *latus rectum* or *parameter.*

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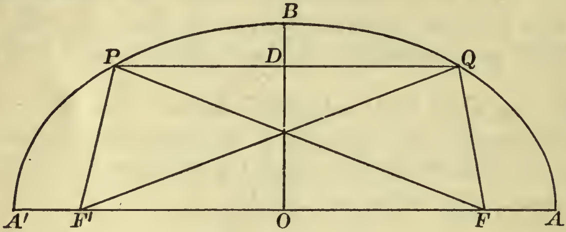
849. Remark. In the following propositions F and F' denote foci of the ellipse, O its centre, AA' the major axis, and BB' the minor axis.

PROPOSITION XII. THEOREM.

850. ***An ellipse is symmetrical with respect to its minor axis.***



Let P be a point of the curve, PDQ be perpendicular to OB, meeting OB in D, and let DQ equal DP.

To prove that Q is also a point of the curve.

Proof. Join P and Q to the foci F, F'.

Revolve $ODQF$ about OD; F will fall on F' and Q on P.

Therefore $QF = PF'$,

and $\angle PQF = \angle QPF'$.

Therefore $\triangle PQF = \triangle QPF'$, § 106

and $QF' = PF$.

Hence $QF + QF' = PF + PF'$.

But $PF + PF' = 2a$. Hyp.

Therefore $QF + QF' = 2a$.

Therefore Q is a point of the curve.

Q. E. D.

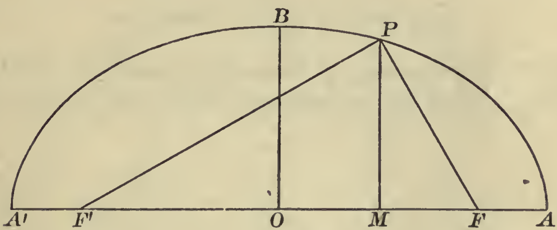
851. Every chord passing through the centre of an ellipse is called a *diameter*.

852. COR. 1. From §§ 847, 850, it follows that *an ellipse consists of four equal quadrantal arcs symmetrically placed about its centre O* (§§ 209, 64).

853. COR. 2. *Every diameter is bisected at the centre.*

Proposition XIII. Theorem.

854. *If d denotes the abscissa of a point of an ellipse, r and r' its focal radii, then $r' = a + ed$, $r = a - ed$.*



Let P be any point of an ellipse, PM perpendicular to AA', d equal OM, r equal PF, r' equal PF'.

To prove $\quad r' = a + ed, \quad r = a - ed.$

Proof. From the rt. △ FPM and $F'PM$

$$r^2 = \overline{PM}^2 + \overline{FM}^2,$$

$$r'^2 = \overline{PM}^2 + \overline{F'M}^2.$$

Therefore $\quad r'^2 - r^2 = \overline{F'M}^2 - \overline{FM}^2.$

Or $\quad (r' + r)(r' - r) = (F'M + FM)(F'M - FM).$

Now $\quad r' + r = 2a$, and $F'M + FM = 2c.$

Also, $\quad F'M - FM = OF' + OM - FM = 2OM = 2d$

Hence $\quad a(r' - r) = 2cd.$

$$r' - r = \frac{2cd}{a} = 2ed.$$

From $\quad r' + r = 2a$, and $r' - r = 2ed$,

$$2r' = 2(a + ed), \text{ and } 2r = 2(a - ed).$$

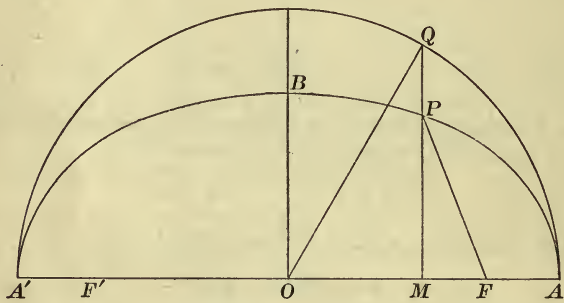
Therefore $\quad r' = a + ed$, and $r = a - ed$.

Q. E. D.

855. The circle described upon the major axis of an ellipse as a diameter is called the *auxiliary circle.* The points where a line perpendicular to the major axis meets the ellipse and its auxiliary circle are called *corresponding points.*

PROPOSITION XIV. THEOREM.

856. *The ordinates of two corresponding points in an ellipse and its auxiliary circle are in the ratio* $b:a$.



Let P be a point of the ellipse, Q the corresponding point of the auxiliary circle, and QP meet AA' at M.

To prove $PM : QM = b : a.$

Proof. Let $OM = d;$

then $\overline{QM}^2 = a^2 - d^2.$

$$\overline{PM}^2 = \overline{PF}^2 - \overline{FM}^2 = (a - ed)^2 - (c - d)^2 \quad \S\ 854$$

$$= a^2 - 2aed + e^2d^2 - c^2 + 2cd - d^2.$$

Or, since $c = ae$ and $a^2 - c^2 = b^2$, § 846

$$\overline{PM}^2 = b^2 - (1 - e^2)d^2 = \frac{b^2}{a^2}(a^2 - d^2).$$

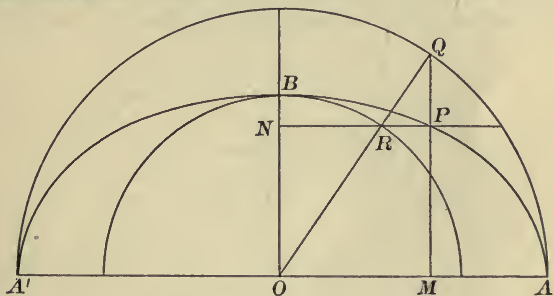
Therefore $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2.$

Or $PM : QM = b : a.$

Q. E. D.

Proposition XV. Problem.

857. *To construct an ellipse by points, having given its two axes.*



Let OA, OB be the given semi-axes, O the centre.

Construction. With O as centre, and OA, OB, respectively, as radii, describe circles.

From O draw any straight line meeting the larger circle at Q and the smaller circle at R.

Through Q draw a line ∥ to OB, and through R draw a line ∥ to OA.

Let these lines meet at P.

Then will P be a point of the required ellipse.

Proof. If QP meet AA' at M,

$$PM : QM = OR : OQ. \qquad \text{§ 309}$$

But $OR = b$ and $OQ = a$.

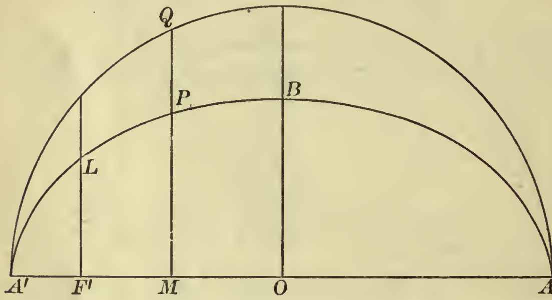
Therefore $PM : QM = b : a$.

Therefore P is a point of the ellipse. (§ 856)

By drawing other lines through O, any number of points on the ellipse may be found; a smooth curve drawn through all the points will be the ellipse required. Q. E. F.

Proposition XVI. Theorem.

858. *The square of the ordinate of a point in an ellipse is to the product of the segments of the major axis made by the ordinate as* $b^2 : a^2$.



Let P, Q be corresponding points in the ellipse and auxiliary circle, respectively; let PQ meet AA' in M.

To prove $\overline{PM}^2 : AM \times A'M = b^2 : a^2$.

Proof. $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2$. § 856

But $\overline{QM}^2 = AM \times A'M$. § 337

Therefore $\overline{PM}^2 : AM \times A'M = b^2 : a^2$. Q. E. D.

859. Cor. *The latus rectum is a third proportional to the major axis and the minor axis.*

For $\overline{LF}^2 : AF \times A'F = b^2 : a^2$. § 858

Now $A'F = a - c$,

and $AF = a + c$.

Therefore $AF \times A'F = a^2 - c^2 = b^2$. § 846

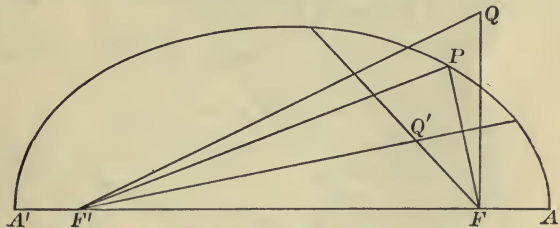
Hence $\overline{LF}^2 : b^2 = b^2 : a^2$,

and $LF : b = b : a$.

Therefore $2a : 2b = 2b : 2LF$.

PROPOSITION XVII. THEOREM.

860. *The sum of the distances of any point from the foci of an ellipse is greater or less than* $2a$, *according as the point is without or within the curve.*



1. Let Q be a point without the curve.

To prove $QF + QF' > 2a$.

Proof. Let P be any point on the arc of the curve between QF and QF'. Draw PF and PF'.

Then $QF + QF' > PF + PF'$. § 118

But $PF + PF' = 2a$. § 838

Therefore $QF + QF' > 2a$.

2. Let Q' be a point within the curve.

To prove $Q'F + Q'F' < 2a$.

Proof. Let P be any point of the curve between FQ' and $F'Q'$ produced.

Then $Q'F + Q'F' < PF + PF'$. § 118

But $PF + PF' = 2a$.

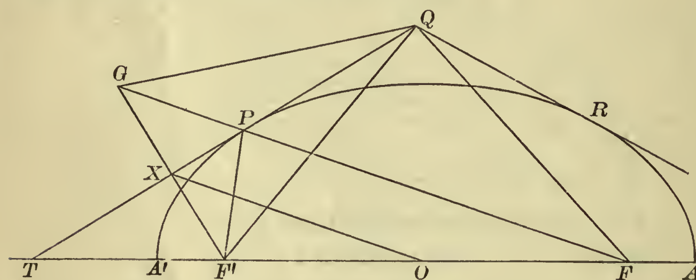
Therefore $Q'F + Q'F' < 2a$. Q. E. D.

861. COR. *Conversely, a point is without or within an ellipse according as the sum of its distances from the foci is greater or less than* $2a$.

862. A straight line which touches but does not cut an ellipse is called a *tangent* to the ellipse. The point where it touches the ellipse is called the *point of contact.*

PROPOSITION XVIII. THEOREM.

863. *If through a point P of an ellipse a line be drawn bisecting the angle between one of the focal radii and the other produced, every point in this line except P is without the curve.*



Let PT bisect the angle F'PG between F'P and FP produced, and let Q be any point in PT except P.

To prove that Q is without the curve.

Proof. Upon FP produced take $PG = PF'$.

Join GF', QF, QF', QG.

Then	$QG + QF > GF'$.	§ 137
Now	$\triangle GPQ = \triangle F'PQ$.	§ 150
Therefore	$QG = QF'$.	
Also	$GF = 2a$.	§ 638
Therefore	$QF' + QF > 2a$.	
Therefore	Q is without the curve.	§ 861

Q. E. D.

864. COR. 1. *PT is the tangent at P.* § 862

865. COR. 2. *The tangent to an ellipse at any point bisects the angle between one focal radius and the other produced.*

866. Cor. 3. *If GF' cuts PT at X, then $GX = F'X$, and PT is perpendicular to GF'.* § 123

867. Cor. 4. *The locus of the foot of a perpendicular dropped from the focus of an ellipse to a tangent is the auxiliary circle.*

For, join OX.

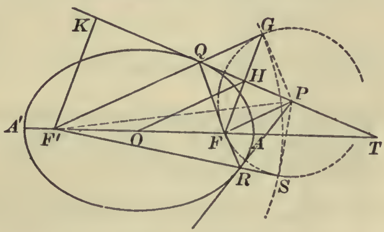
Since $F'X = GX$, and $F'O = OF$,

therefore $OX = \frac{1}{2} FG = \frac{1}{2}(2a) = a$. § 189

Therefore the point X lies in the auxiliary circle.

Proposition XIX. Problem.

868. *To draw a tangent to an ellipse from an exterior point.*



To draw tangents to the ellipse ORQ from the exterior point P.

Construction. Describe arcs with P as centre and PF as radius, and with F' as centre and $2a$ as radius; let these arcs intersect in G and S.

Join GF' and SF', cutting the curve in Q and R respectively.

Join QP and RP, and they will be the tangents required.

Proof. $PG = PF$, and $QG = QF$. Cons., § 838

$\therefore \triangle PQG = \triangle PQF$. § 160

$\therefore \angle PQG = \angle PQF$.

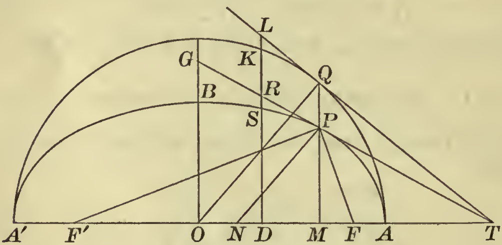
Therefore PQ is the tangent at Q.

For like reason PR is the tangent at R. Q. E. F.

869. Cor. The Ⓢ GFS and GS will always intersect (Ex. 78). Hence, *two tangents may always be drawn to an ellipse from an exterior point.*

Proposition XX. Theorem.

870. *The tangents drawn at two corresponding points of an ellipse and its auxiliary circle cut the major axis produced at the same point.*



Let the tangent to the auxiliary circle at Q cut the major axis produced at T, and let the ordinate QM meet the ellipse at P. Draw PT.

To prove that PT is the tangent to the ellipse at P.

Proof. Through R, any point in PT except P, draw $RD \perp$ to AA', cutting the tangent QT, the auxiliary circle, and the ellipse, in L, K, and S, respectively.

Then $\quad RD : PM = DT : MT = LD : QM,$ § 321

or $\quad RD : LD = PM : QM.$ § 298

But $\quad PM : QM = b : a.$ § 856

$\quad \therefore RD : LD = b : a.$

Again, $\quad SD : KD = b : a.$ § 856

$\quad \therefore RD : LD = SD : KD.$

But $\quad LD > KD.$

$\quad \therefore RD > SD.$

$\therefore R$ is without the ellipse.

Hence PT is the tangent at P. § 862

Q. E. D.

871. Cor. 1. $OT \times OM = a^2$. § 334

872. The straight line PN drawn through the point of contact of a tangent, perpendicular to the tangent, is called the *normal*.

MT is called the *subtangent*, MN the *subnormal*.

873. Cor. 2. *The normal bisects the angle between the focal radii of the point of contact.*

For $\angle TPN = \angle GPN = 90°$. § 872

Subtract $\angle TPF = \angle GPF'$. § 865

And $\angle FPN = \angle F'PN$.

Hence a ray of light issuing from F will be reflected to F'.

874. Cor. 3. *If d denote the abscissa of the point of contact, the distances measured on the major axis from the centre to the tangent and the normal are $\frac{a^2}{d}$ and e^2d, respectively.*

(1) Since $OM = d$, and $OT \times OM = a^2$, § 871

therefore $$OT = \frac{a^2}{d}.$$

(2) Since $OM \times MT = \overline{QM}^2$, § 334

and $MN \times MT = \overline{PM}^2$,

therefore $$\frac{OM}{MN} = \frac{\overline{QM}^2}{\overline{PM}^2} = \frac{a^2}{b^2}.$$

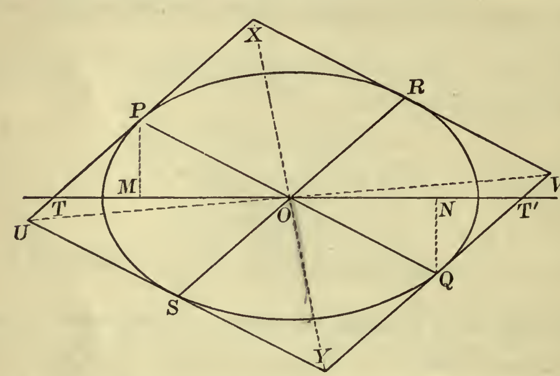
Therefore $$\frac{OM - MN}{OM} = \frac{a^2 - b^2}{a^2} = \frac{c^2}{a^2} = e^2.$$ § 301

That is, $$\frac{ON}{OM} = e^2.$$

Hence $$ON = e^2 \times OM = e^2d.$$

PROPOSITION XXI. PROBLEM.

875. *The tangents drawn at the ends of any diameter are parallel to each other.*



Let POQ be any diameter, PT and QT' the tangents at P, Q respectively, meeting the major axis at T, T'.

To prove $PT \parallel$ to QT'.

Proof. Draw the ordinates PM, QN.

Then $\triangle OPM = \triangle OQN$. § 148

Therefore $OM = ON$.

But $OT = \dfrac{a^2}{OM}$, and $OT' = \dfrac{a^2}{ON}$. § 874

Hence $OT = OT'$.

Therefore $\triangle OPT = \triangle OQT'$, § 150

and $\angle OPT = \angle OQT'$.

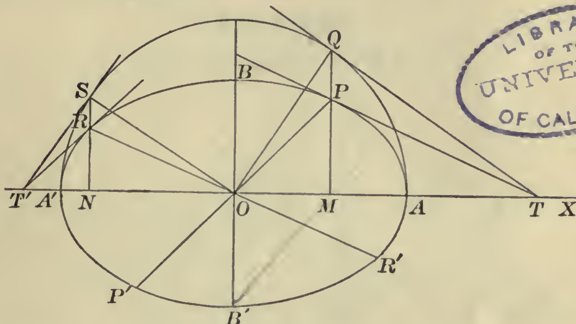
Hence PT is $\parallel$ to QT'. Q. E. D.

876. One diameter is *conjugate* to another, if the first is parallel to the tangents at the extremities of the second.

Thus if ROS is $\parallel$ to PT, RS is conjugate to PQ.

Proposition XXII. Theorem.

877. *If one diameter is conjugate to a second, the second is conjugate to the first.*



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Let the diameter POP' be parallel to the tangent RT'.

To prove that ROR' is parallel to the tangent PT.

Proof. Draw the ordinates PM and RN, and produce them to meet the auxiliary circle in Q and S.

Join OP, OQ, OR, OS; and draw the tangents QT, ST'.

Now, since OP is ∥ to RT',

the △ OMP and $T'NR$ are similar. § 321

$$\therefore T'N : OM = NR : MP.$$

But $NR : NS = MP : MQ$, § 870

or $NR : MP = NS : MQ$.

$$\therefore T'N : OM = NS : MQ.$$

Hence △ $T'NS$ and OMQ are similar. § 326

$$\therefore \angle NT'S = \angle MOQ.$$

$\therefore T'S$ is ∥ to OQ. § 105

Hence $\angle QOS = \angle OST' = 90°$. § 240

$\therefore SO$ is ∥ to QT. § 105

$\therefore \triangle SNO$ and QMT are similar.

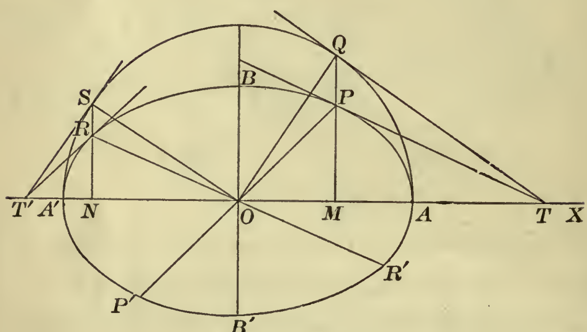
$\therefore ON : TM = NS : MQ,$

$= NR : MP.$ § 856

$\therefore \triangle ONR$ and TMP are similar.

$\therefore OR$ is ∥ to PT.

$\therefore RR'$ is conjugate to PP'. Q. E. D.



878. Cor. 1. *Angle QOS is a right angle.*

879. Cor. 2. $MP : ON = b : a$.

For $OS = OQ,$

and OS is ⊥ to OQ. § 878

Also OM is ⊥ to NS.

$\therefore \angle NSO = \angle MOQ.$ § 113, Rem.

Hence $\triangle NSO = \triangle MOQ.$ § 148

$\therefore ON = MQ.$

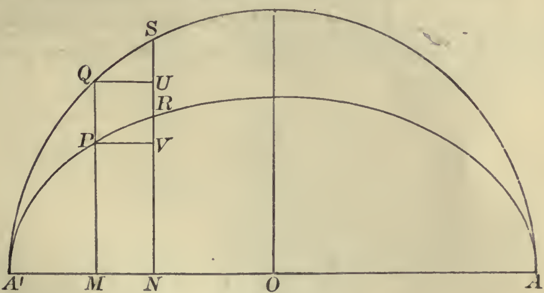
$\therefore MP : ON = MP : MQ.$

But $MP : MQ = b : a.$ § 856

Hence $MP : ON = b : a.$

Proposition XXIII. Theorem.

880. *The area of an ellipse is equal to πab.*



Let A'PRA be any semi-ellipse.

To prove that the area of twice $A'PRA$ is equal to πab.

Proof. Let PM, RN be two ordinates of the ellipse, and let Q, S be the corresponding points on the auxiliary circle.

Draw PV, QU ∥ to the major axis, meeting NS in V, U.

Then $\qquad \square\, PN = PM \times MN,$

and $\qquad \square\, QN = QM \times MN.$

Therefore $\dfrac{\square\, PN}{\square\, QN} = \dfrac{PM \times MN}{QM \times MN} = \dfrac{PM}{QM} = \dfrac{b}{a}.$ § 856

The same relation will be true for all the rectangles that can be similarly inscribed in the ellipse and auxiliary circle.

Hence $\dfrac{\text{sum of ⧈ in ellipse}}{\text{sum of ⧈ in circle}} = \dfrac{b}{a}.$ § 303

And this is true whatever be the number of the rectangles.

But the limit of the sum of the ⧈ in the ellipse is the area of the ellipse, and the limit of those in the ⊙ is the area of the ⊙.

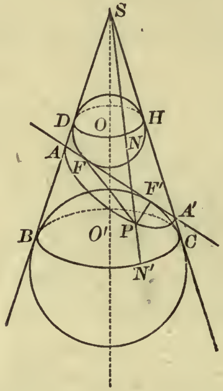
Therefore $\dfrac{\text{area of ellipse}}{\text{area of circle}} = \dfrac{b}{a}.$ § 260

Therefore the area of the ellipse $= \dfrac{b}{a} \times \pi a^2 = \pi ab.$ § 425

Q. E. D.

Proposition XXIV. Theorem.

881. *The section of a right circular cone made by a plane that cuts all the elements of the surface of the cone is an ellipse.*



Let APA' be the curve traced on the surface of the cone SBC by a plane that cuts all the elements of the surface of the cone.

To prove that the curve APA' is an ellipse.

Proof. The plane passed through the axis of the cone and ⊥ to the secant plane APA' cuts the surface of the cone in the elements SB, SC, and the secant plane in the line AA'.

Describe the ⓢ O and O' tangent to SB, SC, AA'. Let the points of contact be D, H, F, and B, C, F', respectively.

Turn BSC and the ⓢ O, O' about the axis of the cone. The lines SB, SC will generate the surface of a right circular cone cut by the secant plane in the curve APA'; and the ⓢ O, O' will generate spheres which touch the cone in the ⓢ DNH, $BN'C$, and the secant plane in the points F, F'.

Let P be any point on the curve APA'. Draw PF, PF'; and draw SP, which touches the ⓢ DH, BC at the points N, N', respectively.

Since PF and PN are tangent to the sphere O, they are tangent to the circle of the sphere made by a plane passing through P, F, and N.

Therefore $PF = PN$. § 246

Likewise $PF' = PN'$.

Hence
$$PF + PF' = PN + PN'$$
$$= NN', \text{ a constant quantity.}$$

Therefore APA' is an ellipse with the points F and F' for foci, and AA' as $2a$. Q. E. D.

882. Cor. *If the secant plane is parallel to the base, the section is a circle, which is a particular case of the ellipse.*

Exercises.

707. Prove that the major axis is the longest chord that can be drawn in an ellipse.

708. If the angle FBF' is a right angle, prove that $a^2 = 2b^2$.

709. To draw a tangent and a normal at a given point of an ellipse.

710. To draw a tangent to an ellipse parallel to a given straight line.

711. Given the foci; it is required to describe an ellipse touching a given straight line.

712. Prove that $\overline{OF}^2 = OT \times ON$. (See figure, page 414.)

713. Prove that $OM : ON = a^2 : c^2$. (See figure, page 414.)

714. The minor axis is the shortest diameter of an ellipse.

715. At what points of an ellipse will the normal at the point pass through the centre of the ellipse?

716. Prove that if FR, $F'S$ are the perpendiculars dropped from the foci to any tangent, then $FR \times F'S = b^2$.

717. To draw a diameter conjugate to a given diameter in a given ellipse.

718. Given $2a$, $2b$, one focus, and one point of the curve, to construct the curve.

719. If from a point P a pair of tangents PQ and PR be drawn to an ellipse, then PQ and PR will subtend equal angles at either focus.

720. To find the foci of an ellipse, having given the major axis and one point on the curve.

721. To find the foci of an ellipse, having given the major axis and a straight line which touches the curve.

722. A straight line moves so that its extremities are always in contact with two fixed straight lines perpendicular to each other. Prove that any point of the moving line describes an ellipse.

723. To construct an ellipse, having given one of the foci and three tangents.

724. To construct an ellipse, having given one focus, two tangents, and one of the points of contact.

725. To construct an ellipse, having given one focus, one vertex, and one tangent.

726. The area of the parallelogram formed by drawing tangents to an ellipse at the extremities of any pair of conjugate diameters is equal to the rectangle contained by the axes of the ellipse.

The Hyperbola.

883. The locus of a point which moves so that the difference of its distances from two fixed points is constant is called an *hyperbola*.

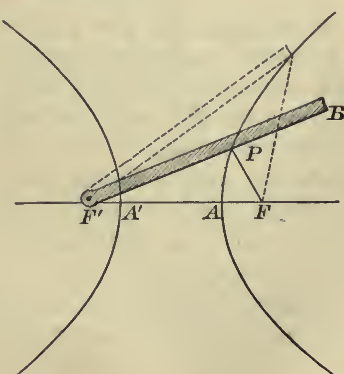
The fixed points are called the *foci*, and the straight lines which join a point of the locus to the foci are called *focal radii*.

The constant difference is denoted by $2a$, and the distance between the foci by $2c$.

The ratio $c:a$ is called the *eccentricity*, and is denoted by e. Therefore $c = ae$.

884. Cor. $2a$ *must be less than* $2c$ (§ 137); *hence* e *must be greater than* 1.

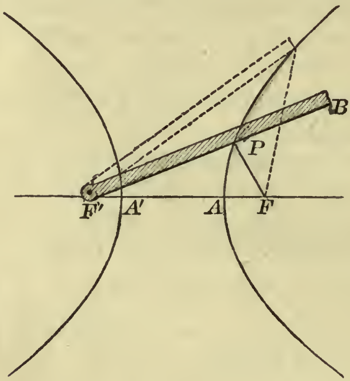
885. An hyperbola may be described by the continuous motion of a point, as follows:



To one of the foci F' fasten one end of a rigid bar $F'B$ so that it is capable of turning freely about F' as a centre in the plane of the paper.

Take a string whose length is less than that of the bar by the constant difference $2a$, and fasten one end of it at the other focus F, and the other end at the extremity B of the bar.

If now the rod is made to revolve about F' while the string is kept constantly stretched by the point of a pencil at P, in contact with the bar, the point P will trace an hyperbola.



For, as the bar revolves, $F'P$ and FP are each increasing by the same amount; namely, the length of that portion of the string which is removed from the bar between any two positions of P; hence the difference between $F'P$ and FP will remain constantly the same.

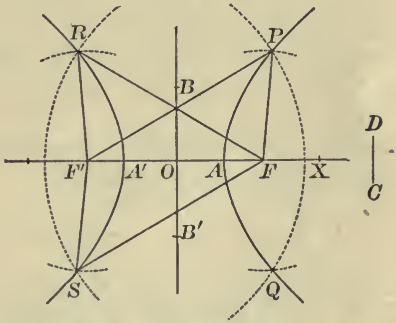
The curve obtained by turning the bar about F' is the right-hand *branch* of the hyperbola. Another similar branch on the left may be described in the same manner by making the bar revolve about F as a centre.

If the two branches of the hyperbola cut the line FF' at A and A', it is easy to see, from the symmetry of the construction, that $AA' = 2a$.

The hyperbola, therefore, is not a closed curve, like the ellipse, but consists of two similar branches which are separated at their nearest points by the distance $2a$, and which recede indefinitely from the line FF' and from one another.

Proposition XXV. Problem.

886. *To construct an hyperbola by points, having given the foci and the constant difference $2a$.*



Let F, F' be the foci, and $a = CD$.

To construct the hyperbola.

Construction. Lay off $OA = OA' = CD$.

Then A and A' are two points of the curve.

For from the construction $AA' = 2a$ and $AF = A'F'$.

Therefore $AF' - AF = AF' - A'F' = AA' = 2a$.

And $A'F - A'F' = A'F - AF = AA' = 2a$.

To locate other points, mark any point X in $F'F$ produced. Describe arcs with F' and F as centres, and $A'X$ and AX as radii, intersecting in P, Q.

Then P, Q are points of the curve.

By describing the same arcs with the foci interchanged, two more points R, S may be found.

By assuming other points in $F'F$ produced, any number of points may be found.

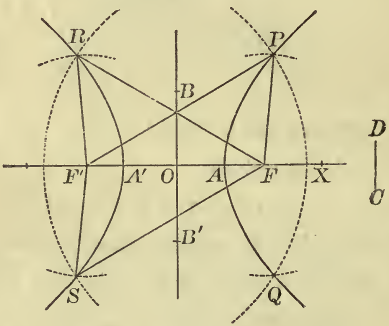
The curve passing through all the points thus determined is an hyperbola having FF' for foci and $2a$ for the constant difference of the focal radii. Q. E. F.

887. Cor. 1. *No point of the curve can be situated on the perpendicular to FF' erected at O.* For every point of this perpendicular is equidistant from the foci.

888. The point O is called the *centre;* AA' is called the *transverse axis;* A and A' are called the *vertices.*

In the perpendicular to FF' erected at O, let B, B' be the two points whose distance from A (or A') is equal to c; then BB' is called the *conjugate axis*, and the length BB' is denoted by $2b$.

If the transverse and conjugate axes are equal, the hyperbola is said to be *equilateral* or *rectangular.*



889. Cor. 2. *Both the axes are bisected at O.*

890. Cor. 3. *It is evident that* $c^2 = a^2 + b^2$.

891. Cor. 4. *The curve is symmetrical with respect to the transverse axis.*

892. The distances of a point of the curve from the transverse and conjugate axes are called respectively the *ordinate* and *abscissa* of the point. The double ordinate through the focus is called the *latus rectum* or *parameter.*

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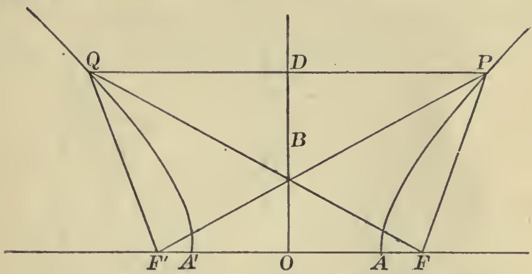
893. Remark. The letters A, A', B, B', F, F', and O, will be used to designate the same points as in the above figure.

PROPOSITION XXVI. THEOREM.

894. *An hyperbola is symmetrical with respect to its conjugate axis.*



Let P be a point of the curve, PDQ be perpendicular to OB, meeting OB at D, and let DQ equal DP.

To prove that Q is also a point of the curve.

Proof. Join P and Q to the foci F, F'.

Turn $ODQF'$ about OD; F' will fall on F, and Q on P.

Therefore $QF' = PF$,

and $\angle PQF' = \angle QPF$.

Therefore $\triangle PQF' = \triangle QPF$, § 150

and $QF = PF'$.

Hence $QF - QF' = PF' - PF$.

But $PF' - PF = 2a$. Hyp.

Therefore $QF - QF' = 2a$.

Therefore Q is a point of the curve. Q. E. D.

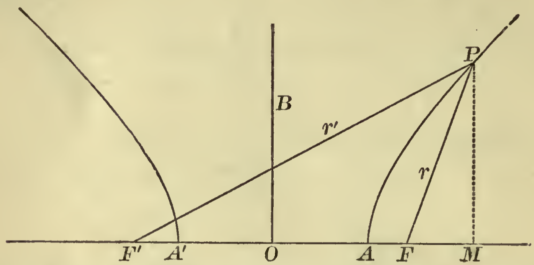
895. Every chord passing through the centre is called a *diameter*.

896. COR. 1. *An hyperbola consists of four equal quadrantal arcs symmetrically placed about its centre O.* § 209

897. COR. 2. *Every diameter is bisected at O.*

PROPOSITION XXVII. THEOREM.

898. *If d denote the abscissa of a point of an hyperbola, r and r' its focal radii, then $r = ed - a$, and $r' = ed + a$.*



Let P be any point of the curve, PM perpendicular to AA', d equal OM, r equal PF, r' equal PF'.

To prove $\quad r = ed - a, \quad r' = ed + a.$

Proof. From the rt. △ FPM, $F'PM$,

$$r^2 = \overline{PM}^2 + \overline{FM}^2,$$

$$r'^2 = \overline{PM}^2 + \overline{F'M}^2.$$

Therefore $\quad r'^2 - r^2 = \overline{F'M}^2 - \overline{FM}^2.$

Or $\quad (r' + r)(r' - r) = (F'M + FM)(F'M - FM).$

Now $\quad r' - r = 2a$, and $F'M - FM = 2c$.

Also $\quad F'M + FM = 2\,OF + 2\,FM = 2\,OM = 2d.$

By substituting these values,

$$a(r' + r) = 2\,cd.$$

Or $$r' + r = \frac{2cd}{a} = 2\,ed.$$

From $\quad r' + r = 2\,ed$, and $r' - r = 2a$,

by addition, $\quad 2r' = 2(ed + a)$;

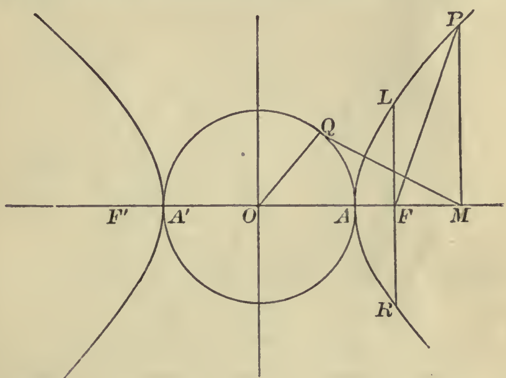
by subtraction, $\quad 2r = 2(ed - a)$.

Therefore $\quad r = ed - a$, and $r' = ed + a$. Q. E. D.

899. The circle described upon AA' as a diameter is called the *auxiliary circle.*

PROPOSITION XXVIII. THEOREM.

900. *Any ordinate of an hyperbola is to the tangent from its foot to the auxiliary circle as b is to a.*



Let P be any point of the hyperbola, PM the ordinate, MQ the tangent drawn from M to the auxiliary circle.

To prove $PM : QM = b : a.$

Proof. Let $OM = d.$

Then $\overline{QM}^2 = d^2 - a^2.$

Also $\overline{PM}^2 = \overline{PF}^2 - \overline{FM}^2$

$= (ed - a)^2 - (d - c)^2$ § 898

$= e^2d^2 - 2aed + a^2 - d^2 + 2cd - c^2.$

Or since $c = ae$, and $a^2 - c^2 = -b^2$, § 890

$$\overline{PM}^2 = (e^2 - 1)\,d^2 - b^2 = \frac{b^2}{a^2}(d^2 - a^2).$$

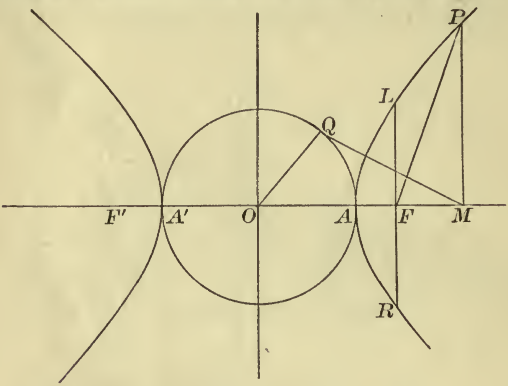
Therefore $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2.$

Or $PM : QM = b : a.$

Q. E. D.

Proposition XXIX. Theorem.

901. *The square of the ordinate of a point in an hyperbola is to the product of the distances from the foot of the ordinate to the vertices as b^2 is to a^2.*



Let P be any point of the curve, PM the ordinate, MQ the tangent drawn from M to the auxiliary circle.

To prove $\overline{PM}^2 : AM \times A'M = b^2 : a^2$.

Proof. Now $\overline{PM}^2 : \overline{QM}^2 = b^2 : a^2$.

But $\overline{QM}^2 = AM \times A'M$. § 348

Therefore $\overline{PM}^2 : AM \times A'M = b^2 : a^2$.

Q. E. D.

902. Cor. *The latus rectum is a third proportional to the transverse and conjugate axes.*

For $\overline{LF}^2 : AF \times A'F = b^2 : a^2$. § 901

But $AF = c - a$, and $AF' = c + a$.

Therefore $AF \times A'F = c^2 - a^2 = b^2$. § 890

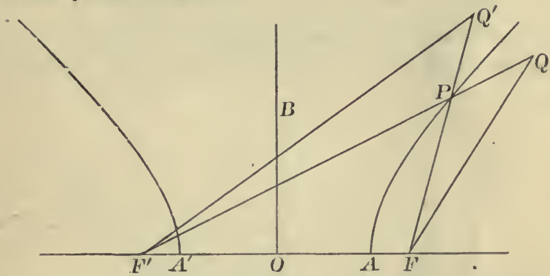
Hence $\overline{LF}^2 : b^2 = b^2 : a^2$.

And $LF : b = b : a$.

Therefore $2a : 2b = 2b : 2LF$.

Proposition XXX. Theorem.

903. *The difference of the distances of any point from the foci of an hyperbola is greater or less than $2a$, according as the point is on the concave or convex side of the curve.*



1. Let Q be a point on the concave side of the curve.

To prove $QF' - QF > 2a.$

Proof. Let QF' meet the curve at P.

Since $F'Q = F'P + PQ$, and $FQ < FP + PQ$,

therefore $F'Q - FQ > F'P - FP.$

But $F'P - FP = 2a.$

Therefore $F'Q - FQ > 2a.$

2. Let Q' be a point on the convex side of the curve

To prove $Q'F' - Q'F < 2a.$

Proof. Let $Q'F$ cut the curve at P.

Since $F'Q' < F'P + PQ'$, and $FQ' = FP + PQ'$,

therefore $F'Q' - FQ' < F'P - FP.$

But $F'P - FP = 2a.$

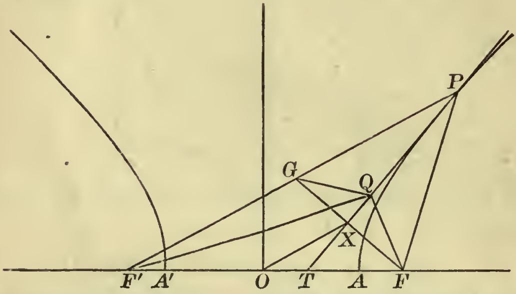
Therefore $F'Q' - FQ' < 2a.$ Q. E. D.

904. Cor. *Conversely, a point is on the concave or the convex side of the hyperbola according as the difference of its distances from the foci is greater or less than $2a$.*

905. A straight line which touches but does not cut the hyperbola is called a *tangent*, and the point where it touches the hyperbola is called the *point of contact*.

Proposition XXXI. Theorem.

906. *If through a point P of an hyperbola a line be drawn bisecting the angle between the focal radii, every point in this line except P is on the convex side of the curve.*



Let PT bisect the angle FPF', and let Q be any point in PT except P.

To prove that Q is on the convex side of the curve.

Proof. On PF' take $PG = PF$; draw FG, QF, QF', QG.

Then $QF' - QG < GF'$, § 137

Also $\triangle PGQ = \triangle PFQ$. § 150

Therefore $QG = QF$.

Also $GF' = PF' - PF = 2a$.

Therefore $QF' - QF < 2a$.

Therefore Q is on the convex side of the curve. § 904

Q. E. D.

907. Cor. 1. *PT is the tangent at P.* § 905

908. Cor. 2. *The tangent to an hyperbola at any point bisects the angle between the focal radii.*

909. Cor. 3. *The tangent at A is perpendicular to AA'.*

910. Cor. 4. *If FG cuts PT at X, then $GX = FX$, and PT is perpendicular to FG.*

911. Cor. 5. *The locus of the foot of the perpendicular from the focus of an hyperbola to a tangent is the auxiliary circle.*

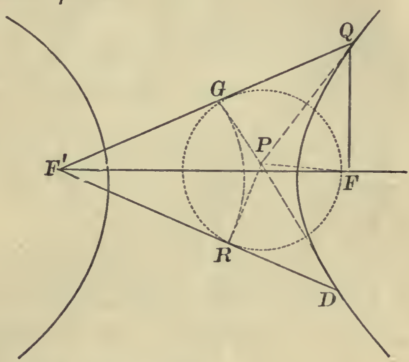
For, since $FX = GX$, and $FO = OF'$,

therefore $OX = \frac{1}{2} F'G = \frac{1}{2}(PF' - PF) = a$. § 189

Therefore the point X lies in the auxiliary circle.

Proposition XXXII. Problem.

912. *To draw a tangent to an hyperbola from a given exterior point.*



Let P be the given point.

Construction. Describe arcs with P as centre and PF as radius, and with F' as centre and $2a$ as radius; let these arcs intersect in G and R.

Draw $F'G$ and $F'R$, and produce them to meet the curve in Q and D, respectively.

Join PQ and PD; PQ and PD are the tangents required.

Proof. $PG = PF$, $QF = QF' - 2a = QG$.

$$\therefore \triangle PQG = \triangle PQF.$$

$$\therefore \angle PQG = \angle PQF.$$

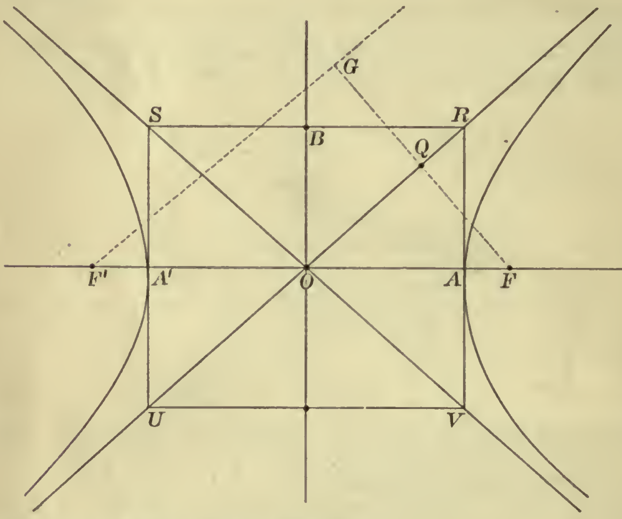
$$\therefore PQ \text{ is the tangent at } Q.$$

For like reason PD is the tangent at D. Q. E. F

913. Cor. *Two tangents may always be drawn to an hyperbola from an exterior point.*

Proposition XXXIII. Theorem.

914. *The tangents to an hyperbola drawn from the centre meet the curve at an infinite distance from the centre.*



Let OR be the tangent from O.

To prove that OR meets the curve at an infinite distance.

Proof. Let G be the intersection of arcs described from O and F' as centres with OF and $2a$ as radii.

The point of contact is the intersection of $F'G$ and OR. § 912

Join FG, cutting OR at Q.

Now $OF' = OF$;

also $QG = QF$. § 910

Therefore $F'G$ is ‖ to OR. § 189

Therefore the point of contact is at an infinite distance.

Q. E. D.

In the same way another tangent OS may be drawn, meeting the other branch of the curve at an infinite distance.

915. The lines OR, OS, indefinitely produced in both directions, are called the *asymptotes* of the hyperbola.

916. Cor. 1. *The line FG is tangent to the auxiliary circle at Q.*

For $\quad FG$ is $\perp$ to OR. § 910

Therefore Q lies on the auxiliary circle. § 911

Hence FG touches the auxiliary circle at Q. § 239

917. Cor. 2. *FQ is equal to the semi-conjugate axis b.*

For $\quad \overline{FQ}^2 = \overline{OF}^2 - \overline{OQ}^2,$ § 339

and $\quad b^2 = c^2 - a^2.$ § 890

But $\quad OF = c$, and $OQ = a$.

Therefore $\quad FQ = b.$

918. Cor. 3. *If the tangent to the curve at A meets the asymptote OR at R, then $AR = b$.*

For $\quad \triangle OAR = \triangle OQF.$ § 149

Therefore $\quad AR = FQ = b.$

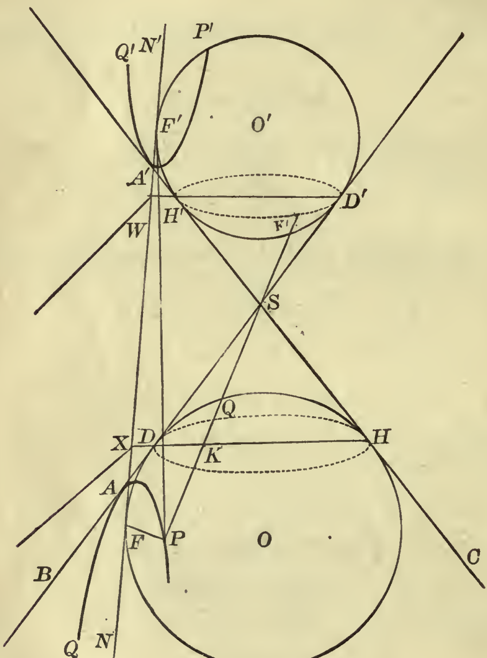
919. Cor. 4. *The asymptotes of an hyperbola are the diagonals of the rectangle $RSUV$ constructed with O for its centre, and the transverse and conjugate axes for its two sides.*

920. A perpendicular to a tangent erected at the point of contact is called a *normal.*

The terms *subtangent* and *subnormal* are used in the hyperbola in the same sense as in the ellipse. § 872

Proposition XXXIV. Theorem.

921. *The section of a right circular cone made by a plane that cuts both nappes of the cone is an hyperbola.*



Let a plane cut the lower nappe of the cone in the curve PAQ, and the upper nappe in the curve $P'A'Q'$.

To prove that PAQ and $P'A'Q'$ are the two branches of an hyperbola.

Proof. The plane passed through the axis of the cone perpendicular to the secant plane cuts the surface of the cone in the elements BS, CS (prolonged through S), and the secant plane in the line NN'.

Describe the ⓢ O, O', tangent to BS, CS, NN'. Let the points of contact be D, H, F, and D', H', F', respectively.

Turn BSC and the ⓢ O and O' about the axis of the cone. BS and CS will generate the surfaces of the two nappes of a right circular cone; and the ⓢ O, O' will generate spheres which touch the cone in the ⓢ DKH, $D'K'H'$, and the secant plane in the points F, F'.

Let P be any point on the curve. Draw PF and PF'; and draw PS, which touches the ⓢ DKH, $D'K'H'$ at the points K, K'.

Now PF and PK are tangents to the sphere O from the point P.

Therefore $$PF = PK.$$

Also $$PF' = PK'.$$

Hence $$PF' - PF = PK' - PK$$
$$= KK', \text{ a constant quantity.}$$

Therefore the curve is an hyperbola with the points F and F' for foci.

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